Extremal Scattered Data Interpolation in $\mathbb{R}^3$ using Triangular Bézier Surfaces

Krassimira Vlachkova

Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski"
Blvd. James Bourchier 5, 1164 Sofia, Bulgaria
krassivl@fmi.uni-sofia.bg

Abstract. We consider the problem of extremal scattered data interpolation in $\mathbb{R}^3$. Using our previous work on minimum L_2-norm interpolation curve networks, we construct a bivariate interpolant F with the following properties:

(i) F is G^1-continuous,
(ii) F consists of triangular Bézier surfaces,
(iii) each Bézier surface satisfies the tetra-harmonic equation $\Delta^4 F = 0$. Hence F is an extremum to the corresponding energy functional.

We also discuss the case of convex scattered data in $\mathbb{R}^3$.

1 Introduction

Scattered data interpolation is a fundamental problem in approximation theory and finds applications in various areas including geology, meteorology, cartography, medicine, computer graphics, geometric modeling, etc. Different methods for solving this problem were applied and reported, excellent surveys can be found in [4,5,6,7].

The problem can be formulated as follows: Given *scattered data* $(x_i, y_i, z_i) \in \mathbb{R}^3$, $i = 1, \ldots, N$, that is points $V_i = (x_i, y_i)$ are different and non-collinear, find a bivariate function F defined in a certain domain D containing points V_i, such that F possesses continuous partial derivatives up to a given order and $F(x_i, y_i) = z_i$. One of the possible approaches to solving the problem is due to Nielson [8]. The method consists of the following three steps:

Step 1. Triangulation. Construct a triangulation T of $V_i,\ i = 1, \ldots N$.

Step 2. Minimum norm network (MNN). The interpolant F and its first order partial derivatives are defined on the edges of T so as to satisfy an extremal property. The MNN is a cubic curve network, i. e. on every edge of T it is a cubic polynomial.

Step 3. Interpolation surface. The obtained network is extended to F by an appropriate *blending method.* The interpolant F is a rational function on every triangle of T.

In [1] Andersson et al. pay special attention to the second step of the above method - the construction of the MNN. Using a different approach, the authors give a new proof of Nielson's result. Their approach allows to consider and handle the case where the data are convex and we seek a convex interpolant. Andersson et al. formulate the corresponding extremal constrained interpolation problem of finding a MNN that is convex along the edges of the triangulation. The extremal network is characterized as a solution of a nonlinear system of equations. The authors propose a Newton-type algorithm for solving this type of systems. The validity and convergence of the algorithm were studied further in [11].

In this paper we focus on Step 3 of the discussed approach. Since the MNN is a polynomial curve network it is natural to require that the interpolant F also is a polynomial on every triangle of T. Although the MNN is C^1-continuous at the vertices of T, it is preferable and more appropriate to require G^1-continuity for the interpolant instead of C^1-continuity since the latter is parametrization dependent. Two surfaces with a common boundary curve are called *G^1-continuous* if they have a continuously varying tangent plane along that boundary curve.

Let D be the union of all triangles in T. For simplicity we assume that D contains no holes. For given data and the corresponding MNN we construct an interpolation surface $F(u, v)$ defined on D with the following properties:

(i) F consists of triangular Bézier surfaces (patches) defined on each triangle of T;
(ii) F is G^1-continuous;
(iii) F satisfies the tetra-harmonic equation $\Delta^4\mathbf{x} = 0$ a.e. for $(u, v) \in D$, where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the Laplace operator. Hence F is a solution to the extremal problem
$$\min_F \int_D \|\Delta^4 F\|_2 du dv \tag{1}$$
i.e. F is an extremum to the corresponding energy functional.

We also consider the constrained interpolation problem. In this case, despite that the MNN is edge convex, in general it is not globally convex. Therefore a convex interpolation surface may not exist. We propose the following approach. We modify if necessary the edge convex MNN to obtain an edge convex polynomial curve network with the same tangent planes at the vertices of T. Then we construct a G^1-continuous interpolation surface $\widehat{F}(u, v)$ still satisfying (iii).

The paper is organized as follows: In Section 2 we introduce the notation and present some related results from [1]. In Section 3 we investigate the G^1-continuity conditions for adjacent Bézier patches and prove that they correctly apply to our problem. The construction of surface F is considered in Section 4. In the final Section 5 we briefly discuss the constrained interpolation problem.

2 Preliminaries

Let $N \geq 3$ be an integer and $(x_i, y_i, z_i),\ i = 1, \ldots, N$ be given scattered data. Points $V_i := (x_i, y_i)$ are the projection points of the data onto the plane Oxy. A *triangulation* T of points V_i is a collection of non-overlapping, non-degenerate closed triangles in Oxy such that the set of the vertices of the triangles coincides with the set of points V_i. For a given triangulation T there is a unique continuous function $L : D \to \mathbb{R}^1$ that is linear inside each of the triangles of T and interpolates the data. Scattered data in D are *convex* if there exists a triangulation T of V_i such that the corresponding function L is convex. The data are *strictly convex* if they are convex and the gradient of L has a jump discontinuity across each edge inside D.

The set of the edges of the triangles in T is denoted by E. If there is an edge between V_i and V_j in E, it will be referred to by e_{ij} or simply by e if no ambiguity arises. A *curve network* is a collection of real-valued univariate functions $\{f_e\}_{e\in E}$ defined on the edges in E. With any real-valued bivariate function F defined on D we naturally associate the curve network defined as the restriction of F on the edges in E, i.e. for $e = e_{ij} \in E$,

$$f_e(t) := F\left((1-\frac{t}{\|e\|})x_i + \frac{t}{\|e\|}x_j,\ (1-\frac{t}{\|e\|})y_i + \frac{t}{\|e\|}y_j\right), \qquad (2)$$
$$\text{where } 0 \le t \le \|e\|, \text{ and } \|e\| = \sqrt{(x_i-x_j)^2+(y_i-y_j)^2}.$$

Furthermore, according to the context F will denote either a real-valued bivariate function or a curve network defined by (2). We introduce the following class of functions defined on D

$$\mathcal{F} := \{F(x,y) \mid F(x_i,y_i) = z_i,\ i = 1,\dots,N,\ \partial F/\partial x, \partial F/\partial y \in C(D),$$
$$f'_e \in AC_{[0,\|e\|]}, f''_e \in L^2_{[0,\|e\|]}, e \in E\}$$

and the corresponding classes of *smooth interpolation curve networks*

$$\mathcal{C}(E) := \left\{F_{|E} = \{f_e\}_{e\in E} \mid F(x,y) \in \mathcal{F},\ e \in E\right\}$$

and *smooth interpolation edge convex curve networks*

$$\widehat{\mathcal{C}}(E) := \left\{F_{|E} = \{f_e\}_{e\in E} \mid F(x,y) \in \mathcal{F},\ f''_e \ge 0,\ e \in E\right\}.$$

For $F \in \{\mathcal{C}(E), \widehat{\mathcal{C}}(E)\}$ we denote the curve network of second derivatives of F by $F'' := \{f''_e\}_{e\in E}$. The L_2-norm of F'' is defined by

$$\|F''\|_{L_2(T)} := \|F''\| = \left(\sum_{e\in E}\int_0^{\|e\|} |f''_e(t)|^2 dt\right)^{1/2}.$$

We consider the following two extremal problems.

$$(\mathbf{P}) \qquad \textit{Find } F^* \in \mathcal{C}(E) \ \textit{ such that } \|F^{*\prime\prime}\| = \inf_{F\in\mathcal{C}(E)} \|F''\|,$$

$$(\widehat{\mathbf{P}}) \qquad \textit{Find } \widehat{F}^* \in \widehat{\mathcal{C}}(E) \ \textit{ such that } \|\widehat{F}^{*\prime\prime}\| = \inf_{F\in\widehat{\mathcal{C}}(E)} \|F''\|.$$

In [1,8] it has been shown that $(\mathbf{P})$ and $(\widehat{\mathbf{P}})$ for strictly convex data have unique solutions.

3 The G^1-continuity conditions

3.1 Control points that are next to a boundary curve

Let C_1 and C_2 be cubic triangular Bézier patches with a common boundary which is cubic polynomial $q(t)$. Let $q(t) = \sum_{i=0}^{3} \mathbf{q}_i B_i^3(t)$ where $\mathbf{q}_i,\ i = 0, \dots, 3$ are the control points of $q(t)$, and $B_i^n(t)$ are the Bernstein polynomials of degree n, $n \in \mathbb{N}$, defined for $0 \le t \le 1$ as follows:

$$B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i},$$

$$\binom{n}{i} = \begin{cases} \frac{n!}{i!(n-i)!} & \text{if } 0 \le i \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{p}_i$ and $\mathbf{r}_i$, $i = 0, \dots, 3$ are next to the boundary control points of C_1 and C_2, respectively. Let us degree elevate $q(t)$ to quartic polynomial and denote the degree elevated control points by $\hat{\mathbf{q}}_i,\ i = 0, \dots, 4$, where $\hat{\mathbf{q}}_0 \equiv \mathbf{q}_0$ and $\hat{\mathbf{q}}_4 \equiv \mathbf{q}_3$. Then $q(t) = \sum_{i=0}^{4} \hat{\mathbf{q}}_i B_i^4(t)$ where

$$\hat{\mathbf{q}}_i = \frac{i}{4}\mathbf{q}_{i-1} + \left(1 - \frac{i}{4}\right)\mathbf{q}_i,\ i = 0, \dots, 4. \tag{3}$$

Farin [3] proposed the following sufficient conditions for G^1-continuity between C_1 and C_2.

$$\frac{i}{4} d_{i,4} + \left(1 - \frac{i}{4}\right) d_{i,0} = 0,\ i = 0, \dots, 4, \text{ where} \tag{4}$$

$$\begin{aligned} d_{i,0} &= \alpha_0 \mathbf{p}_i + (1-\alpha_0)\mathbf{r}_i - (\beta_0 \hat{\mathbf{q}}_i + (1-\beta_0)\hat{\mathbf{q}}_{i+1}), \\ d_{i,4} &= \alpha_1 \mathbf{p}_{i-1} + (1-\alpha_1)\mathbf{r}_{i-1} - (\beta_1 \hat{\mathbf{q}}_{i-1} + (1-\beta_1)\hat{\mathbf{q}}_i), \end{aligned}$$

and $0 < \alpha_i < 1,\ i = 1, 2$. Next we shall prove that system (4) always has a solution. From (4) for $i = 0$ and $i = 4$ we obtain

$$d_{0,0} = 0 \Rightarrow \alpha_0 \mathbf{p}_0 + (1-\alpha_0)\mathbf{r}_0 = \beta_0 \hat{\mathbf{q}}_0 + (1-\beta_0)\hat{\mathbf{q}}_1, \tag{5}$$

$$d_{4,4} = 0 \Rightarrow \alpha_1 \mathbf{p}_3 + (1-\alpha_1)\mathbf{r}_3 = \beta_1 \hat{\mathbf{q}}_3 + (1-\beta_1)\hat{\mathbf{q}}_4 \tag{6}$$

The points $\hat{\mathbf{q}}_0$, $\hat{\mathbf{q}}_1$, $\mathbf{p}_0$, and $\mathbf{r}_0$ are coplanar since they lie on the tangent plane at $\hat{\mathbf{q}}_0$, see Fig. 2(a). Hence α_0 and β_0 are uniquely determined from (5) by the intersection point of the diagonals of the planar quadrilateral $\hat{\mathbf{q}}_0 \mathbf{r}_0 \hat{\mathbf{q}}_1 \mathbf{p}_0$. Note that the quadrilateral could be non-convex and in this case either $\beta_0 > 1$, or $\beta_0 < 0$. Analogously, α_1 and β_1 are uniquely determined by (6). Therefore system (4) has three equations and four unknowns $\mathbf{p}_1, \mathbf{r}_1, \mathbf{p}_2, \mathbf{r}_2$ as follows.

$$\left| \begin{array}{l} \alpha_0 \mathbf{p}_1 + (1-\alpha_0)\mathbf{r}_1 = c_1 \\ \alpha_1 \mathbf{p}_1 + (1-\alpha_1)\mathbf{r}_1 + \alpha_0 \mathbf{p}_2 + (1-\alpha_0)\mathbf{r}_2 = c_2 \\ \alpha_1 \mathbf{p}_2 + (1-\alpha_1)\mathbf{r}_2 = c_3 \end{array} \right. , \tag{7}$$

where

$$\begin{aligned}
c_1 &:= \beta_0\hat{\mathbf{q}}_1 + (1-\beta_0)\hat{\mathbf{q}}_2 - \tfrac{1}{3}(\alpha_1\mathbf{p_0} + (1-\alpha_1)\mathbf{r}_0 - \beta_1\hat{\mathbf{q}}_0 - (1-\beta_1)\hat{\mathbf{q}}_1),\\
c_2 &:= \beta_0\hat{\mathbf{q}}_2 + (1-\beta_0)\hat{\mathbf{q}}_3 + \beta_1\hat{\mathbf{q}}_1 + (1-\beta_1)\hat{\mathbf{q}}_2,\\
c_3 &:= \beta_1\hat{\mathbf{q}}_2 + (1-\beta_1)\hat{\mathbf{q}}_3 - \tfrac{1}{3}(\alpha_0\mathbf{p}_3 + (1-\alpha_0)\mathbf{r}_3 - \beta_0\hat{\mathbf{q}}_3 - (1-\beta_0)\hat{\mathbf{q}}_4).
\end{aligned} \tag{8}$$

We shall prove the following

Lemma 1. *System* (7) *always has a solution.*

Proof. If $\alpha_0 \neq \alpha_1$ then the rank of matrix M of system (7) is 3 and the system always has a solution. If $\alpha_0 = \alpha_1$ we have

$$0 = c_1 - c_2 + c_3 = \frac{1}{3}(\beta_1 - \beta_0)\sum_{i=0}^{4}(-1)^i\binom{4}{i}\hat{\mathbf{q}}_i. \tag{9}$$

Thus rank(M)=2 and the system is compatible if and only if the right-hand side of (9) is zero. This holds obviously for $\beta_0 = \beta_1$. If $\beta_0 \neq \beta_1$ we use (3) and obtain consecutively

$$\begin{aligned}
&\textstyle\sum_{i=0}^{4}(-1)^i\binom{4}{i}\hat{\mathbf{q}}_i = \sum_{i=1}^{4}(-1)^i\binom{4}{i}\frac{i}{4}\mathbf{q}_{i-1} + \sum_{i=0}^{3}(-1)^i\binom{4}{i}\frac{4-i}{4}\mathbf{q}_i\\
&= \textstyle\sum_{i=1}^{4}(-1)^i\binom{3}{i-1}\mathbf{q}_{i-1} + \sum_{i=0}^{3}(-1)^i\binom{3}{i}\mathbf{q}_i\\
&= \textstyle\sum_{i=0}^{3}(-1)^{i+1}\binom{3}{i}\mathbf{q}_i + \sum_{i=0}^{3}(-1)^i\binom{3}{i}\mathbf{q}_i = 0.
\end{aligned}$$

□

3.2 The vertex enclosure problem

Let $\mathbf{q}_0$ be an inner vertex in T with $\deg\mathbf{q}_0 = n$ where the *degree* is the number of the edges incident to $\mathbf{q}_0$. For $k = 1,\ldots,n$ let Q_k be the quartic Bézier patches with common vertex $\mathbf{q}_0$; $q^k(t) = \sum_{i=0}^{4}\hat{\mathbf{q}}_i^k B_i^4(t)$ be the degree elevated cubic curves of the MNN emanating from $\mathbf{q}_0$ with the corresponding α_i^k, β_i^k, $i = 0,1$; and $\mathbf{t}_k$ be next to $\mathbf{q}_0$ inner control point of Q_k, see Fig. 1 for $n = 4$. We apply (5) to Q_k and obtain the following linear system for the unknowns $\mathbf{t}_k$,

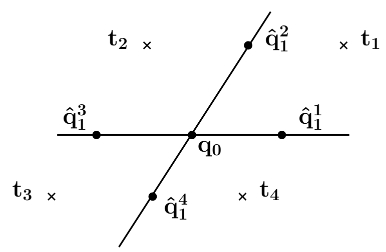

Fig. 1. The vertex enclosure problem: points $\mathbf{t}_i$, $i = 1,\ldots,4$ must satisfy a linear system of equations

$$\begin{pmatrix}
1-\alpha_0^2 & \alpha_0^2 & \ldots & 0 & 0\\
0 & 1-\alpha_0^3 & \alpha_0^3 & \ldots & 0\\
\vdots & & & & \\
0 & 0 & \ldots & 1-\alpha_0^n & \alpha_0^n\\
\alpha_0^1 & 0 & \ldots & 0 & 1-\alpha_0^1
\end{pmatrix}
\begin{pmatrix}\mathbf{t}_1\\ \mathbf{t}_2\\ \vdots\\ \mathbf{t}_{n-1}\\ \mathbf{t}_n\end{pmatrix}
=
\begin{pmatrix}\mathrm{s}_1\\ \mathrm{s}_2\\ \vdots\\ \mathrm{s}_{n-1}\\ \mathrm{s}_n,\end{pmatrix}, \tag{10}$$

where

$$
\begin{aligned}
\mathrm{s}_i &= \beta_0^{i+1}\hat{\mathbf{q}}_1^{i+1} + (1-\beta_0^{i+1})\hat{\mathbf{q}}_2^{i+1} - \tfrac{1}{3}(\alpha_1^{i+1}\hat{\mathbf{q}}_1^{i+2} + (1-\alpha_1^{i+1})\hat{\mathbf{q}}_1^{i} \\
&\quad -\beta_1^{i+1}\hat{\mathbf{q}}_0 - (1-\beta_1^{i+1})\hat{\mathbf{q}}_1^{i+1}),\ i = 1,\dots,n-2, \\
\mathrm{s}_{n-1} &= \beta_0^{n}\hat{\mathbf{q}}_1^{n} + (1-\beta_0^{n})\hat{\mathbf{q}}_2^{n} - \tfrac{1}{3}(\alpha_1^{n}\hat{\mathbf{q}}_1^{1} + (1-\alpha_1^{n})\hat{\mathbf{q}}_1^{n-1} - \beta_1^{n}\hat{\mathbf{q}}_0 - (1-\beta_1^{n})\hat{\mathbf{q}}_1^{n}), \\
\mathrm{s}_n &= \beta_0^{1}\hat{\mathbf{q}}_1^{1} + (1-\beta_0^{1})\hat{\mathbf{q}}_2^{1} - \tfrac{1}{3}(\alpha_1^{1}\hat{\mathbf{q}}_1^{2} + (1-\alpha_1^{1})\hat{\mathbf{q}}_1^{n} - \beta_1^{1}\hat{\mathbf{q}}_0 - (1-\beta_1^{1})\hat{\mathbf{q}}_1^{1}).
\end{aligned}
$$

The determinant of (10) is

$$\mathrm{Det} = (-1)^{n+1}\alpha_0^1\alpha_0^2\dots\alpha_0^n + (1-\alpha_0^1)(1-\alpha_0^2)\dots(1-\alpha_0^n).$$

It can be verified that $\dfrac{(1-\alpha_0^1)(1-\alpha_0^2)\dots(1-\alpha_0^n)}{\alpha_0^1\alpha_0^2\dots\alpha_0^n} = 1$ and therefore we have $\mathrm{Det} = \begin{cases} 2\alpha_0^1\alpha_0^2\dots\alpha_0^n, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$. The latter implies

Lemma 2. *The rank of the matrix of system (10) is* $\begin{cases} n, & n \textit{ odd}, \\ n-1, & n \textit{ even}. \end{cases}$

Therefore if n is odd then system (10) always has a solution, and if n is even then (10) could be incompatible and may not have a solution. The existence of a solution to system (10) is known as the *vertex enclosure problem*, see [10].

4 Construction of the Bézier patches

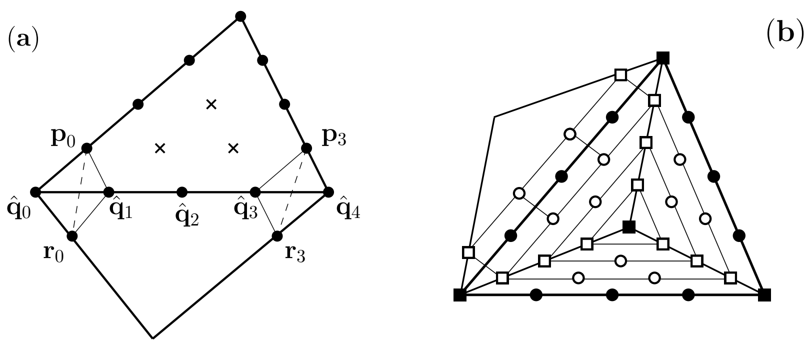

Fig. 2. Construction of the G^1-continuous Bézier patches: (a) without splitting; (b) splitting the patch into three sub-patches

In this section we show how to construct successfully G^1-continuous Bézier patches in the presence of even degree vertices. We apply procedure called *splitting* similar to the one proposed by Clough and Tocher, see [2,9]. Let T^1 denote the set of triangles in T that have an inner vertex of even degree. We *split* all triangles in T^1. Let $\tau \in T^1$. We partition τ into three new sub-triangles with a common vertex in τ, see Fig. 2(b). This new vertex with appropriate z-value is added to the data. In [10] it is shown that splitting leads to admissible data. Further, computing the new vertex in τ is done through computation of control points into three sub-triangles obtaining in this way the three G^1-continuous Bézier sub-patches. Algorithm 1 below takes a triangle τ in T^1 and degree-elevated quartic boundary control points of the corresponding patch and computes the control points of the three G^1-continuous Bézier sub-patches.

Algorithm 1: Splitting - Fig. 2(b)

Input: Degree elevated quartic boundary control points ■,● of a Bézier patch
Output: Control points of three G^1- continuous interpolating sub-patches
Step 1. Find the first layer of inner control points that are next to the boundary:
1.1 Points □ are centers of the three triangles with vertices ●■●.
1.2 Then points ○ divide segments □□ into three equal parts.
Step 2. Find the second layer of inner control points:
2.1 Points □ are centers of the three small triangles with vertices ○□○ from the first layer.
2.2 Then points ○ are midpoints of segments □□.
Step 3. Find the third layer of inner control points: Three points □ are centers of the small triangles with vertices ○□○ from the second layer.
Step 4. Find the last inner control point ■ as the center of the triangle with vertices □ from the third layer.

For a triangle in $T \setminus T^1$ (see Fig. 2(a)) we have $\alpha_0 = \alpha_1$ and $\beta_0 = \beta_1$ since the projections of the quadrilaterals $\hat{\mathbf{q}}_0\mathbf{r}_0\hat{\mathbf{q}}_1\mathbf{p}_0$ and $\hat{\mathbf{q}}_3\mathbf{r}_3\hat{\mathbf{q}}_4\mathbf{p}_3$ onto the plane Oxy are congruent. We compute the three inner control points marked by $\times$, from the corresponding three systems (10) for each of the triangle vertices. Two of the equations in systems (10) are the same as in system (7) for the G^1-continuity conditions. The third equation in (7) is

$$(\alpha_1 - \alpha_0)(\mathbf{r}_1 - \mathbf{p}_1 - \mathbf{r}_2 + \mathbf{p}_2) = \frac{1}{3}(\beta_1 - \beta_0)\sum_{i=0}^{4}(-1)^i\binom{4}{i}\hat{\mathbf{q}}_i.$$

It is automatically satisfied since $\alpha_0 = \alpha_1$ and the right-hand side is zero according to the proof of Lemma 1.

Using Algorithm 1 we construct G^1-continuous surface $F(u,v)$ defined on D which consists of triangular quartic Bézier patches and interpolates the MNN. The next theorem states the extremal properties of F.

Theorem 1. *$F(u,v)$ satisfies the tetra-harmonic equation $\Delta^4\mathbf{x} = 0$ for $(u,v) \in D \setminus E$. Consequently F is a solution to the extremal problem (1) and hence F is an extremum to the energy functional $\Phi(\mathbf{x}) = 1/2\int_D \|\Delta^4\mathbf{x}\|^2 dudv$.*

5 The constrained extremal problem

Let the given data be strictly convex. In this case $\widehat{F}^*$ on every edge is either a convex cubic polynomial or a convex cubic spline with one knot, see [1]. If $\hat{f}_e^*$ is a polynomial we do not modify it. If $\hat{f}_e^*$ is a spline, we slightly modify it to obtain a convex cubic curve with the same tangents at the ends. Then, using the results from Section 2 and Section 3 we construct G^1-continuous surface $\widehat{F}$ defined on D which consists of triangular quartic Bézier patches, interpolates the modified edge convex MNN, and still satisfies Theorem 1. The main difficulty in this case is in the algorithm for splitting since the projections of the control points onto the corresponding edge are not equally spaced. The details will be presented in the full version.

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