Extremal Interpolation of Convex Scattered Data in $\mathbb{R}^3$ Using Tensor Product Bézier Surfaces

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Abstract. We consider the problem of extremal interpolation of convex scattered data in $\mathbb{R}^3$ and propose a feasible solution. Using our previous work on edge convex minimum L_p-norm interpolation curve networks, $1 < p \leq \infty$, we construct a bivariate interpolant F with the following properties:

(i) F is G^1-continuous;
(ii) F consists of tensor product Bézier surfaces (patches) of degree (n, n) where $n \in \mathbb{N}$, $n \geq 4$, is priorly chosen;
(iii) The boundary curves of each patch are convex;
(iv) Each Bézier patch satisfies the tetra-harmonic equation $\Delta^4 F = 0$. Hence F is an extremum to the corresponding energy functional.

1 Introduction

Scattered data interpolation is a fundamental problem in approximation theory and finds applications in various areas including geology, meteorology, cartography, medicine, computer graphics, geometric modeling etc. Different methods for solving this problem were applied and reported, excellent surveys are [4,5,8,9].

The problem can be formulated as follows: Given *scattered data* $\mathbf{d}_i = (x_i, y_i, z_i) \in \mathbb{R}^3$, $i = 1, \ldots, N$, that is points $\mathbf{v}_i = (x_i, y_i)$ are different and non-collinear, find a bivariate function F defined in a certain domain D containing points $\mathbf{v}_i$, such that F possesses continuous partial derivatives up to a given order and $F(x_i, y_i) = z_i$. One of the possible approaches to solving the problem is due to Nielson [14]. The method consists of the following three steps:

Step 1. Triangulation. Construct a triangulation T of $\mathbf{v}_i$, $i = 1, \ldots N$.

Step 2. Minimum norm network (MNN). The interpolant F and its first order partial derivatives are defined on the edges of T so as to satisfy an extremal property.

Step 3. Interpolation surface. The obtained network is extended to F by an appropriate *blending method.*

In [1] Andersson et al. paid special attention to the second step of the above method, i.e. the construction of the MNN. The authors applied a novel approach and gave an alternative proof of Nielson's result. Their method allows to consider and handle the case where the data are convex and a convex interpolant is sought. Andersson et al. formulated the corresponding extremal constrained interpolation problem of finding a MNN that is convex along the edges of the triangulation. The extremal network was characterized as a solution to a nonlinear system of equations and a Newton-type algorithm for solving this type of systems was proposed. The results from [1] are extended in [16] to the class of L_p-norms for $1 < p \leq \infty$. The validity and convergence of the Newton-type algorithm for $1 < p \leq \infty$ were studied further in [17]. We note that the edge convex MNN may not be globally convex and hence a convex interpolation surface may not exist at all. Moreover, even in the case where the edge convex MNN is globally convex, Nielson's blending method may produce non-convex surface.

In this paper we propose the following solution to the convex scattered interpolation problem. Instead of triangulation we construct a rectilinear quadrangulation Q having points $\mathbf{v}_i = (x_i, y_i)$, $i = 1, \dots, N$, as its vertices, see Fig. 1. We define suitable z-values for different from $\mathbf{v}_i$ vertices of Q (if any) and add the new points to our data. Then we compute the edge convex minimum L_p-norm network for $p = \frac{n-1}{n-2}$ where $n \in \mathbb{N}$, $n \geq 3$, is chosen in advance. Hereafter we assume that n is part of our input data. The obtained edge convex MNN on every edge of Q is either a polynomial of degree n or a C^1-continuous spline with one inner knot consisting of a linear function plus a polynomial of degree n, see [16]. Moreover, the obtained network is not only edge convex but also it is convex on every whole row or column of Q. This is one of the reasons we use rectilinear quadrangulation instead of triangulation. Despite that, in general the edge convex MNN still may not be globally convex. For that reason we are seeking to construct an interpolation surface that is computationally simple and minimizes some appropriately chosen energy functional. Surfaces with such properties tend to preserve convexity of the input data. Nielson's blending method [13,14] produces an interpolant which is a rational function on every triangle in T and consecutively may have large values in terms of energy. So, our idea is as follows. First, we slightly modify the edge-convex MNN on the edges where it is a spline. The modified MNN is C^1-continuous with the same tangent planes at the vertices of Q, consists of edge convex Bézier curves of degree n, and is convex on every row or column of Q. Next, we find a piecewise polynomial surface that interpolates the modified MNN and minimizes an appropriate energy functional. Although the modified MNN is C^1-continuous at the vertices of T, it is preferable and more appropriate to require G^1-continuity for the interpolant instead of C^1-continuity since the latter is parametrization dependent. We recall that two surfaces with a common boundary curve are G^1-*continuous* if they have a continuously varying tangent plane along that boundary curve.

Let D be the union of all quadrangles in Q. For simplicity we assume that D contains no holes. We construct a surface $F(u, v)$ defined on D that interpolates the modified MNN and has the following properties:

(i) F consists of tensor product Bézier surfaces (patches) of degree (n, n). Each patch is defined on a quadrangle of the mesh;
(ii) F is G^1-continuous;
(iii) The boundary curves of each patch are convex;

(iv) F satisfies the tetra-harmonic equation $\Delta^4 F = 0$ a.e. for $(u, v) \in D$ where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the Laplace operator. Hence F is a solution to the extremal problem

$$\operatorname*{argmin}_{\mathbf{x}\in\mathcal{F}} \int_D \|\Delta^4 \mathbf{x}\| du dv, \tag{1}$$

where $\mathcal{F} := \left\{ \mathbf{x}(u, v) \, : \, \mathbf{x}(\mathbf{v}_i) = z_i, \; i = 1, \ldots, N, \; \mathbf{x} \in W_2^8(D) \right\}$, and $W_2^8(D)$ is the corresponding Sobolev space. Then F is an extremum to the corresponding energy functional.

The harmonic and bi-harmonic Bézier surfaces were studied by Monterde and Ugail [11]. Their method was extended to general 4th-order PDE Bézier surfaces in [12]. Here we use a result by Centella et al. [2] to generate tetra-harmonic tensor product Bézier surfaces from given boundary curves and tangent conditions along them. The corresponding unconstrained problem for scattered data interpolation in $\mathbb{R}^3$ is considered and solved in [18] using a rectangular quadrangulation.

The paper is organised as follows. In Sect. 2 we introduce the notation, present some related results from [1,16], and propose our Algorithm 1 for solving the convex scattered data interpolation problem. In Sect. 3 we discuss the construction of the surface F.

2 Preliminaries and Description of the Algorithm

A *quadrangulation* of given points in $\mathbb{R}^2$ is a collection of non-overlapping, non-degenerate closed quadrangles such that the set of the vertices of the quadrangles coincides with the set of the points. We shall assume that our points $\mathbf{v}_i$, $i = 1, \ldots, N$, are vertices of a quadrangulation Q that is homeomorphic to a rectilinear quadrangulation where vertical (horizontal) lines are not necessarily parallel. Given set of points in a general position in $\mathbb{R}^2$ one can not construct a rectilinear quadrangulation having these points as its vertices. However, we can construct it so that all of our points are among its vertices. The remaining vertices are added to the given points, see Fig. 1. An obvious way is to draw vertical and horizontal lines through each of our points. We can also choose appropriately two directions in the plane and draw lines parallel to the chosen directions through each of $\mathbf{v}_i$, $i = 1, \ldots, N$. Clearly this approach does not lead to unique quadrangulation. It is an open question to find the dependance of the input surface on the initial choice of the quadrangulation. Furthermore, the above approach has a drawback that the number of the new vertices is quadratic in terms of N. Our

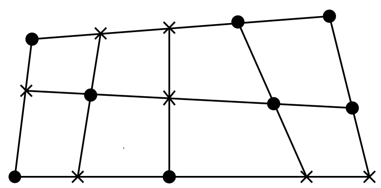

Fig. 1: Rectilinear quadrangulation of the projection points $\mathbf{v}_i$, $i = 1, \ldots, N$, where $\bullet$ denotes old (given) points, and $\times$ denotes new (added) points.

method works directly in the case where Q is a rectilinear quadrangulation, see Fig. 1. The benefit of constructing a rectilinear quadrangulation so that points $\mathbf{v}_i$,, $i = 1, \ldots, N$, are its vertices is that the number of the new (added) vertices would be reduced considerably although in the general case their number still would be quadratic in terms of N. Given scattered data finding an optimal in terms of size rectilinear quadrangulation is beyond the scope of this paper.

We define z_i-values for the new points so that the data are in a convex position. The latter is possible due to the following lemma from [1].

Lemma 1. ([1]) *If the data are convex (strictly convex) then there exists a convex (strictly convex) function* $\psi \in C^\infty(\mathbb{R}^2)$ *interpolating the points* $\mathbf{d}_i$, $i = 1, \ldots, N$.

The proof of Lemma 1 is constructive and the function ψ is constructed in a polynomial time. We choose the z_i-values for the new points $\mathbf{d}_i$ so that $\psi(x_i, y_i) = z_i$. Hereafter we suppose that our data are strictly convex.

The union of all quadrangles in Q is the domain D. The set of the edges of the quadrangles in Q is denoted by E. If there is an edge between $\mathbf{v}_i$ and $\mathbf{v}_j$ in E, it will be referred to by e_{ij} or simply by e if no ambiguity arises. A *curve network* is a collection of real-valued univariate functions $\{f_e\}_{e\in E}$ defined on the edges in E. With any real-valued bivariate function F defined on D we naturally associate the curve network defined as the restriction of F on the edges in E, i.e. for $e = e_{ij} \in E$,

$$
\begin{aligned}
f_e(t) := F\Big(\big(1 - \frac{t}{\|e\|}\big)x_i + \frac{t}{\|e\|}\,x_j,\ \big(1 - \frac{t}{\|e\|}\big)y_i + \frac{t}{\|e\|}\,y_j\Big),\\
\text{where } 0 \le t \le \|e\| \text{ and } \|e\| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}.
\end{aligned}
\tag{2}
$$

In our presentation, according to the context, F will denote either a real-valued bivariate function or a curve network defined by (2). Let $1 < p < \infty$. We introduce the following class of functions defined on D

$$
\begin{aligned}
\mathcal{F}_p := \big\{F(x,y) \in C^1(D)\ :\ F(x_i, y_i) = z_i,\ i = 1, \ldots, N,\\
f_e' \in AC[0, \|e\|],\ f_e'' \in L^p[0, \|e\|],\ e \in E\big\},
\end{aligned}
$$

and the corresponding class of *smooth interpolation edge convex curve networks*

$$
\mathcal{C}_p(E) := \big\{F|_E = \{f_e\}_{e\in E}\ :\ F(x,y) \in \mathcal{F}_p,\ f_e'' \ge 0,\ e \in E\big\}.
$$

For $F \in \mathcal{C}_p(E)$ we denote the curve network of second derivatives of F by $F'' := \{f_e''\}_{e\in E}$. The L_p-norm of F'' is defined by

$$
\|F''\|_p := \left(\sum_{e\in E} \int_0^{\|e\|} |f_e''(t)|^p dt\right)^{1/p}.
$$

We consider the following extremal problem.

$$
(\mathbf{P}_p) \qquad \textit{Find } F^* \in \mathcal{C}_p(E) \ \textit{ such that } \ \|F^{*\prime\prime}\|_p = \inf_{F\in\mathcal{C}_p(E)} \|F''\|_p.
$$

The degree of all inner vertices in Q, i.e. the number of the edges in E incident to each inner vertex, is four. Let $\{e_{ii_1}, \dots, e_{ii_4}\}$ be the edges incident to the inner vertex $\mathbf{v}_i$ listed in clockwise order around $\mathbf{v}_i$. A *basic curve network* B_{is} is defined on E for $s = 1, 2$ as follows.

$$B_{is} := \begin{cases} 1 - \frac{t}{\|e_{ii_{s+r}}\|} & \text{on } e_{ii_{s+r}},\ 0 \le t \le \|e_{ii_{s+r}}\|,\ r = 0, 2, \\ 0 & \text{on the other edges of } E. \end{cases}$$

Note that basic curve networks are associated with vertices that have at least two collinear edges incident to them. Thus, one basic curve network is associated with each vertex on the boundary of Q except the four corner vertices. We denote by N_B the set of pairs of indices is for which a basic curve network is defined. With each basic curve network B_{is} for $is \in N_B$ we associate a number d_{is} defined by $d_{is} = (z_{i_s} - z_i)/\|e_{ii_s}\| + (z_{i_{s+2}} - z_i)/\|e_{ii_{s+2}}\|$.

The next theorem characterizes the solution to problem $(\mathbf{P}_p)$.

Theorem 1 ([1,16]). *In the case of strictly convex data the problem* $(\mathbf{P}_p)$, $1 < p < \infty$, *has a unique solution* F^*. *The second derivative of the solution* $F^{*\prime\prime}$ *has the form*

$$F^{*\prime\prime} = \left(\sum_{is \in N_B} \alpha_{is} B_{is} \right)_+^{q-1}$$

where $1/p + 1/q = 1$, $(x)_+ := \max(x, 0)$ *and the coefficients* α_{is} *satisfy the following nonlinear system of equations*

$$\int_E \left(\sum_{is \in N_B} \alpha_{is} B_{is} \right)_+^{q-1} B_{kl} dt = d_{kl}, \ \textit{for } kl \in N_B. \tag{3}$$

The basic curve networks B_{is} are the univariate basic B-splines defined along every row and column of the quadrangulation Q and the numbers d_{is} are the univariate second-order divided differences. Our data are strictly convex which guarantees that d_{is} are strictly positive and therefore Theorem 1 applies. The solution to $(\mathbf{P}_p)$ decomposes to $n_1 + n_2$ solutions to the problem in the univariate case along every row and column of Q, where n_1, n_2 are the numbers of the rows and columns of Q respectively, and $n_1 n_2 = N$. In the univariate case the problem of finding a convex function which interpolates given convex data and minimises the energy functional is considered e. g. by Hornung [6] for $p = 2$, and for $1 < p < \infty$ by Iliev and Pollul [7], Micchelli et al. [10].

It follows from Theorem 1 that in the case where $q \in \mathbb{N}$, $q > 1$, then F^* is a C^1-continuous polynomial network and the degree of the polynomials is $n = q+1$. Hence to obtain a polynomial solution to $(\mathbf{P}_p)$ of degree n we solve the nonlinear system (3) for $p = \frac{q}{q-1} = \frac{n-1}{n-2}$ using the Newton-type algorithm [17]. Note that although Theorem 1 holds for $1 < p < \infty$, we apply it only for $1 < p \le 2$ since $n \ge 3$. Further on, we consider the polynomials in its Bézier form and propose Algorithm 1 for solving the convex scattered data extremal

Algorithm 1 Extremal Convex Scattered Data Interpolation

Input: Strictly convex scattered data $\mathbf{d}_i = (x_i, y_i, z_i) \in \mathbb{R}^3$, $i = 1, \dots, N$; $n \in \mathbb{N}$, $n \geq 3$.
Output: Interpolation surface F with certain extremal property
Step 1. Construct quadrangulation Q of the projection points $\mathbf{v}_i = (x_i, y_i)$, $i = 1, \dots, N$, using straight lines through them.
Step 2. Add new input points to the data if necessary.
Step 3. Solve $(\mathbf{P}_p)$ for $p = \frac{n-1}{n-2}$.
Step 4. Construct the modified edge convex MNN.
4.1 Compute the control points of the modified curves (if any).
4.2 Degree elevate all curves to curves of degree $n + 1$.
Step 5. For each quadrangle in Q find nearest to the boundary control points that satisfy G^1 continuity conditions.
Step 6. Find the remaining inner control points so that the tensor product Bézier surface for each quadrangle satisfies the tetra-harmonic equation $\Delta^4 F = 0$.

interpolation problem. Step 5 and Step 6 of Algorithm 1 are similar to the corresponding steps of the algorithm for the unconstrained case proposed in [18] where they are considered in detail. In the next Section 3 we focus mainly on Step 4 - the construction of the modified edge convex MNN.

3 Construction of the Bézier Patches

Let B_1 and B_2 be tensor product Bézier patches whose common boundary is the polynomial $q(t)$ of degree n, $n \in \mathbb{N}$. First, we consider sufficient conditions for G^1 continuity between B_1 and B_2. Let $q(t) = \sum_{i=0}^{n} \mathbf{q}_i B_i^n(t)$ where $\mathbf{q}_i$, $i = 0, \dots, n$, are the control points of $q(t)$, and $B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i}$, $i = 0, \dots, n$, are the Bernstein polynomials defined for $0 \leq t \leq 1$. We degree elevate $q(t)$ to a polynomial of degree $n+1$. Then $q(t) = \sum_{i=0}^{n+1} \hat{\mathbf{q}}_i B_i^{n+1}(t)$ where $\hat{\mathbf{q}}_i$, $i = 0, \dots, n+1$, are the degree elevated control points. Let $\mathbf{p}_i$ and $\mathbf{r}_i$, $i = 0, \dots, n$, be nearest to the boundary control points of B_1 and B_2, respectively. Farin [3] proposed the following sufficient conditions for G^1 continuity between B_1 and B_2:

$$\frac{i}{n+1} d_{i,n+1} + \Big(1 - \frac{i}{n+1}\Big) d_{i,0} = 0, \; i = 0, \dots, n+1, \text{ where} \tag{4}$$

$$\begin{aligned} d_{i,0} &= \alpha_0 \mathbf{p}_i + (1-\alpha_0)\mathbf{r}_i - \big(\beta_0 \hat{\mathbf{q}}_i + (1-\beta_0)\hat{\mathbf{q}}_{i+1}\big), \\ d_{i,n+1} &= \alpha_1 \mathbf{p}_{i-1} + (1-\alpha_1)\mathbf{r}_{i-1} - \big(\beta_1 \hat{\mathbf{q}}_{i-1} + (1-\beta_1)\hat{\mathbf{q}}_i\big), \end{aligned}$$

and $0 < \alpha_0, \alpha_1 < 1$. The coefficients α_0 and α_1 are uniquely determined by the intersection point of segments $\mathbf{p}_0\mathbf{r}_0$, $\hat{\mathbf{q}}_0\hat{\mathbf{q}}_1$, and $\mathbf{p}_n\mathbf{r}_n$, $\hat{\mathbf{q}}_n\hat{\mathbf{q}}_{n+1}$, respectively. In [18] it is shown that in the case where $\alpha_0 = \alpha_1$, system (4) always has a solution. The *vertex enclosure problem* is also solved since we use a rectilinear quadrangulation, see [15,18] for details.

To construct an interpolating Bézier patch, the boundary curves need to be polynomial curves. For that reason, we first modify the edge convex MNN so that all edge curves comprising it are Bézier curves of degree n. In the case where f_e^* for some $e \in E$ is a spline (see Fig. 2a for $n = 3$) we modify it to the Bézier curve of degree n whose Bézier polygon has the same graph as the Bézier polygon of the spline, see Fig. 2b for $n = 3$. The modified curve is convex and has the same tangents at the endpoints as the spline.

a. **b.**

Fig. 2: Case $n = 3$: **a.** Convex C^1-continuous spline with one inner knot and its control polygon. The spline consists of a linear function plus a cubic function. **b.** The modified cubic convex Bézier curve and its control polygon.

Next, consider a pair of Bézier curves defined on same line neighbouring edges of Q, see Fig. 3. Let $\mathbf{d}_i$ be their common point and s_1, s_2 be the two segments of their Bézier polygons with common endpoint $\mathbf{d}_i$. By construction s_1 and s_2 are collinear. We shorten the longer segment by moving its endpoint towards $\mathbf{d}_i$ so that the new segments s_1 and s_2 become equal. Then we replace the corresponding Bézier polygon by the modified one. In this way we ensure that $\alpha_0 = \alpha_1$ for every edge of Q and hence system (4) can be solved. Then we compute nearest to the boundary control points, see [18] for details.

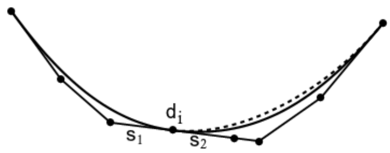

Fig. 3: Pair of cubic Bézier curves defined on same line neighbouring edges of Q. The right curve has been modified. The modified curve is shown dashed.

To compute the rest of the control points for each tensor product Bézier patch B_i we use a result by Centella et al. [2]. It states that given the boundary control points and those adjacent to them of an $(n+1) \times (n+1)$ net there exists a unique tetra-harmonic Bézier surface whose control net has those points as boundary control points and those adjacent to them. Finally, using Algorithm 1 we construct surface $F(u, v)$ defined on D which consists of tensor product Bézier patches of degree (n, n). The surface F interpolates the data since it interpolates the modified edge convex MNN. The next theorem states main properties of F.

Theorem 2. *The interpolant $F(u, v)$ is G^1-continuous and is a solution to the extremal problem* (1).

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