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Interpolation of Convex Scattered data in $\mathbb{R}^3$ Using Edge Convex Minimum L_∞-norm Networks

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Abstract. We consider the extremal problem of interpolation of convex scattered data in $\mathbb{R}^3$ by smooth edge convex curve networks with minimal L_p-norm of the second derivative for $1 < p \leq \infty$. The problem for $p = 2$ was set and solved by Andersson et al. (1995). Vlachkova (2019) extended the results in (Andersson et al., 1995) and solved the problem for $1 < p < \infty$. The minimum edge convex L_p-norm network for $1 < p < \infty$ is obtained from the solution to a system of nonlinear equations with coefficients determined by the data. The solution in the case $1 < p < \infty$ is unique for strictly convex data. The approach used in (Vlachkova, 2019) can not be applied to the corressponding extremal problem for $p = \infty$. In this case the solution is not unique. Here we establish the existence of a solution to the extremal interpolation problem for $p = \infty$. This solution is a quadratic spline function with at most one knot on each edge of the underlying triangulation. We also propose sufficient conditions for solving the problem for $p = \infty$.

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INTRODUCTION

Interpolation of scattered data in $\mathbb{R}^3$ is an important problem in applied mathematics and finds applications in various fields such as computer graphics and animation, scientific visualization, medicine (computer tomography), automotive, aircraft and ship design, architecture, and many more. In general the problem can be formulated as follows: Given a set of points $P_i = (x_i, y_i, z_i) \in \mathbb{R}^3$, $i = 1, \ldots, n$, find a bivariate function $F(x, y)$ defined in a certain domain D containing points V_i, such that F possesses continuous partial derivatives up to a given order and $F(x_i, y_i) = z_i$. Various methods for solving this problem were proposed and applied, see [1, 2, 3, 4, 5, 6, 7, 8]. Nielson [9] proposed a three steps method for solving the problem as follows:

Step 1. Triangulation. Construct a triangulation T of $V_i,\ i = 1, \ldots n$.

Step 2. Minimum norm network. The interpolant F and its first order partial derivatives are defined on the edges of T to satisfy an extremal property. The obtained minimum norm network is a cubic curve network, i. e. on every edge of T it is a cubic polynomial.

Step 3. Interpolation surface. The obtained network is extended to F by an appropriate *blending method*.

Andersson et al. [10] focused on Step 2 of the above method, namely the construction of the minimum norm network. The authors gave a new proof of Nielson's result by using a different approach. They constructed a system of simple linear curve networks called *basic curve networks* and then represented the second derivative of the minimum norm network as a linear combination of these basic curve networks. The new approach allows to consider and handle the case where the data are convex and we seek a convex interpolant. Andersson et al. formulate the corresponding extremal constrained interpolation problem of finding a minimum norm network that is convex along the edges of the triangulation. The extremal network is characterized as a solution to a nonlinear system of equations. The authors propose a Newton-type algorithm for solving this type of systems. The validity and convergence of the algorithm were studied further in [11].

Vlachkova [12] extended the results in [10] and solved the extremal unconstrained problem of interpolation of scattered data by minimum L_p-norms networks for $1 < p < \infty$. The minimum L_p-norm network for $1 < p < \infty$ is obtained from the solution to a system of nonlinear equations with coefficients determined by the data. The solution in the case $1 < p < \infty$ is unique. The approach proposed in [12] can not be applied to the case where $p = \infty$. In this case, the solution is not unique. Recently, Vlachkova [13] established the existence of a solution for $p = \infty$ of the same type as in the case where $1 < p < \infty$.

The extremal constrained interpolation problem for $1 < p < \infty$ was considered in [14] where the existence and the uniqueness of the solution in the case of strictly convex data were established and a complete characterization of the solution using the basic curve networks was presented. The approach used in [14] does not apply in the case where $p = \infty$. In this case, the solution is also not unique.

In this paper we consider the extremal constrained interpolation problem for $p = \infty$ and show the existence of a solution of a certain type. More precisely, the solution on each edge of the underlying triangulation T is a quadratic spline function with at most one knot. We also establish a theorem that provides sufficient conditions for solving the extremal constrained interpolation problem for $p = \infty$.

PRELIMINARIES AND RELATED WORK

Let $n \geq 3$ be an integer and $P_i := (x_i, y_i, z_i),\ i = 1, \dots, n$ be different points in $\mathbb{R}^3$. We call this set of points *data*. The data are *scattered*[1] if the projections $V_i := (x_i, y_i)$ onto the plane Oxy are different and non-collinear.

Definition 1 *A collection of non-overlapping, non-degenerate triangles in $\mathbb{R}^2$ is a triangulation of the points $V_i,\ i = 1, \dots, n$, if the set of the vertices of the triangles coincides with the set of the points $V_i,\ i = 1, \dots, n$.*

For a given triangulation T there is a unique continuous function $L : D \to \mathbb{R}^1$ that is linear inside each of the triangles of T and interpolates the data.

Definition 2 *Scattered data in D are convex if there exists a triangulation T of V_i such that the corresponding function L is convex. The data are strictly convex if they are convex and the gradient of L has a jump discontinuity across each edge inside D.*

Hereafter we assume that the data are convex and T is an associated triangulation of the points $V_i,\ i = 1, \dots, n$. The set of the edges of the triangles in T is denoted by E. If there is an edge between V_i and V_j in E, it will be referred to by e_{ij} or simply by e if no ambiguity arises.

Definition 3 *A curve network is a collection of real-valued univariate functions $\{f_e\}_{e \in E}$ defined on the edges in E.*

With any real-valued bivariate function F defined on D we naturally associate the curve network defined as the restriction of F on the edges in E, i. e. for $e = e_{ij} \in E$,

$$f_e(t) := F\Big(\big(1 - \frac{t}{\|e\|}\big)x_i + \frac{t}{\|e\|}\,x_j,\ \big(1 - \frac{t}{\|e\|}\big)y_i + \frac{t}{\|e\|}\,y_j\Big),$$
$$\text{where } 0 \leq t \leq \|e\| \text{ and } \|e\| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}. \tag{1}$$

Furthermore, according to the context F will denote either a real-valued bivariate function or a curve network defined by (1). For p, such that $1 < p \leq \infty$, we introduce the following class of *smooth interpolants*

$$\mathscr{F}_p := \{F(x,y) \in C^1(D)\,|\,F(x_i, y_i) = z_i,\ i = 1, \dots, n,\ f_e' \in AC,\ f_e'' \in L_p,\ e \in E\},$$

and the corresponding class of so-called *smooth interpolation edge convex curve networks*

$$\mathrm{C}_p(E) := \left\{F_{|E} = \{f_e\}_{e \in E}\ |\ F(x,y) \in \mathscr{F}_p,\ f_e'' \geq 0,\ e \in E\right\},$$

where $C^1(D)$ is the class of bivariate functions defined in D which possess continuous first order partial derivatives, AC is the class of univariate absolutely continuous functions defined in $[0, \|e\|]$, L_p for $1 < p < \infty$ is the class of univariate functions defined in $[0, \|e\|]$ whose p-th power of the absolute value is Lebesgue integrable, and L_∞ is the class of the bounded univariate functions defined in $[0, \|e\|]$. The smoothness of the interpolation curve network $F = \{f_e\}_{e \in E} \in \mathrm{C}_p(E)$ geometrically means that at each point P_i there is a *tangent plane* to F, where a plane is *tangent*

[1] Note that this definition of scattered data slightly misuses the commonly accepted meaning of the term. It allows data with some structure among points $\mathbf{v}_i$. We have opted to do this in order to cover all cases where our presentation and results are valid.

to the curve network at the point P_i if it contains the tangent vectors at P_i of the curves incident to P_i. Further, L_p-norm are defined in $\mathrm{C}_p(E)$ by

$$\|F\|_p := \left(\sum_{e\in E}\int_0^{\|e\|}|f_e(t)|^p dt\right)^{1/p}, \quad 1<p<\infty,$$

$$\|F\|_\infty := \max_{e\in E}\ \|f_e\|_\infty.$$

We denote the networks of the second derivative of F by $F'' := \{f_e''\}_{e\in E}$ and consider the following extremal problem:

$$(\mathbf{P}_p) \quad \textit{Find } F^* \in \mathrm{C}_p(E) \ \textit{such that } \|F^{*\prime\prime}\|_p = \inf_{F\in \mathrm{C}_p(E)} \|F''\|_p.$$

Problem (P_p) is a generalization of the classical univariate extremal problem $(\tilde{P}_p)$ for interpolation of convex data in $\mathbb{R}^2$ by a univariate convex function with minimal L_p-norm of the second derivative. Hornung [15] considered the problem $(\tilde{P}_p)$ for $p=2$. Iliev and Pollul [16] considered the case $p=\infty$ and proved that problem $(\tilde{P}_\infty)$ has a quadratic spline solution characterized by the existence of a core interval on which the second derivative is the positive part of a perfect spline and that all solutions agree on core intervals. Iliev and Pollul [17], and Micchelli et al. [18] studied in detail the problem for $1<p<\infty$.[2]

For $i=1,\dots,n$, let m_i denote the degree of the vertex V_i, i. e. the number of the edges in E incident to V_i. Furthermore, let $\{e_{ii_1},\dots,e_{ii_{m_i}}\}$ be the edges incident to V_i listed in clockwise order around V_i. The first edge e_{ii_1} is chosen so that the coefficient $\lambda_{1,i}^{(s)}$ defined below is not zero - this is always possible. A *basic curve network* B_{is} is defined on E for any pair of indices is, such that $i=1,\dots,n$ and $s=1,\dots,m_i-2$, as follows:

$$B_{is} := \begin{cases} \lambda_{r,i}^{(s)}\left(1-\frac{t}{\|e_{ii_{s+r-1}}\|}\right) & \text{on } e_{ii_{s+r-1}},\ r=1,2,3, \\ & \quad 0\le t\le \|e_{ii_{s+r-1}}\| \\ 0 & \text{on the other edges of } E. \end{cases}$$

The coefficients $\lambda_{r,i}^{(s)}$, $r=1,2,3$, are uniquely determined to sum to one and to form a zero linear combination of the three unit vectors along the edges $e_{ii_{s+r-1}}$ starting at V_i. Note that basic curve networks are associated with points that have at least three edges incident to them. We denote by N_B the set of pairs of indices is for which a basic curve network is defined, i. e., $N_B := \{is \mid m_i \ge 3,\ i=1,\dots,n,\ s=1,\dots,m_i-2\}$.

With each basic curve network B_{is} for $is\in N_B$ we associate a number d_{is} defined by

$$d_{is} = \frac{\lambda_{1,i}^{(s)}}{\|e_{ii_s}\|}(z_{i_s}-z_i) + \frac{\lambda_{2,i}^{(s)}}{\|e_{ii_{s+1}}\|}(z_{i_{s+1}}-z_i) + \frac{\lambda_{3,i}^{(s)}}{\|e_{ii_{s+2}}\|}(z_{i_{s+2}}-z_i),$$

which reflects the position of the data in the supporting set of B_{is}.

In [14] a full characterization of the solution to the extremal problem (P_p) for $1<p<\infty$ was provided. Its finding reduces to the unique solution of a system of equations. The following theorem was established in [14].

Theorem 1 *In the case of strictly convex data problem (P_p), $1<p<\infty$, has a unique solution F^*. The second derivative of the solution $F^{*\prime\prime}$ has the form*

$$F^{*\prime\prime} = \left(\sum_{is\in N_B}\alpha_{is}B_{is}\right)_+^{q-1}$$

where $1/p+1/q=1$, $(x)_+ := \max(x,0)$ and the coefficients α_{is} satisfy the following nonlinear system of equations

$$\int_E \left(\sum_{is\in N_B}\alpha_{is}B_{is}\right)_+^{q-1} B_{kl}dt = d_{kl},\ \textit{for } kl\in N_B.$$

[2] Iliev and Pollul [17], and Micchelli et al. [18] considered the more general problem of minimum L_p-norm of the k-th derivative which is non-negative, $k\ge 2$.

MAIN RESULTS

Our main results are presented in the next two theorems. Theorem 2 establishes the existence of a solution to problem (P_∞) of a certain type. Theorem 3 provides sufficient conditions for solving (P_∞). It states that if there exists a solution of the same type as in the case where $1 < p < \infty$ (Theorem 1) then it solves problem (P_∞).

Theorem 2 *Problem (P_∞) has a solution whose restriction on every edge $e \in E$ is a quadratic spline with at most one knot in the interval $(0, \|e\|)$.*

Theorem 3 *Let $F_\infty \in C_\infty(E)$ be such that*

$$F''_\infty = c\left(\sum_{is\in N_B} \alpha_{is} B_{is}\right)^0_+$$

where α_{is}, $is \in N_B$, are real numbers, and $c > 0$. Then F_∞ solves problem (P_∞).

Remark 1 *Coefficients α_{is} are determined up to a constant factor.*

Corollary 1 *Coefficients c and α_{is}, $is \in N_B$, are a solution to the system*

$$\int_E c\left(\sum_{is\in N_B} \alpha_{is} B_{is}\right)^0_+ B_{kl} = d_{kl},\ kl \in N_B. \qquad (2)$$

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