# Convergence of the minimum $L_p$-norm networks as $p \to \infty$

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# Convergence of the Minimum $L_p$-Norm Networks as $p \to \infty$

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**Abstract.** We consider the extremal problem of interpolation of scattered data in $\mathbb{R}^3$ by smooth curve networks with minimal $L_p$-norm of the second derivative for $1 < p \leq \infty$. The problem for $p = 2$ was set and solved by Nielson [1]. Andersson et al. [2] gave a new proof of Nielson's result by using a different approach. Vlachkova [3] extended the results in [2] and solved the problem for $1 < p < \infty$. The minimum $L_p$-norm network for $1 < p < \infty$ is obtained from the solution to a system of nonlinear equations with coefficients determined by the data. The solution in the case $1 < p < \infty$ is unique. We denote the corresponding minimum $L_p$-norm network by $F_p$. In the case where $p = \infty$ we establish the existence of a solution of the same type as in the case where $1 < p < \infty$. This solution on each edge of the underlying triangulation is a quadratic spline function with at most one knot. We denote this solution by $F_\infty$ and prove that the minimum $L_p$-norm networks $F_p$ converge to the minimum $L_\infty$-norm network $F_\infty$ as $p \to \infty$.
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## INTRODUCTION

Interpolation of scattered data in $\mathbb{R}^3$ is an important problem in applied mathematics and finds applications in various fields such as automotive, aircraft and ship design, architecture, archeology, medicine (computer tomography), computer graphics and animation, scientific visualization, and many more. In general the problem can be formulated as follows: Given a set of points $P_i = (x_i, y_i, z_i) \in \mathbb{R}^3$, $i = 1, \ldots, n$, find a bivariate function $F(x, y)$ defined in a certain domain $D$ containing points $V_i$, such that $F$ possesses continuous partial derivatives up to a given order and $F(x_i, y_i) = z_i$. Various methods for solving this problem were proposed and applied, see [4, 5, 6, 7, 8, 9, 10, 11, 12]. Nielson [1] proposed a three steps method for solving the problem as follows:

*Step 1. Triangulation*. Construct a triangulation $T$ of $V_i$, $i = 1, \ldots n$.

*Step 2. Minimum norm network*. The interpolant $F$ and its first order partial derivatives are defined on the edges of $T$ to satisfy an extremal property. The obtained minimum norm network is a cubic curve network, i. e. on every edge of $T$ it is a cubic polynomial.

*Step 3. Interpolation surface*. The obtained network is extended to $F$ by an appropriate *blending method*.

Andersson et al. [2] gave a new proof of Nielson's result by using a different approach. They construct a system of simple linear curve networks called *basic curve networks* and then represent the second derivative of the minimum norm network as a linear combination of these basic curve networks. Vlachkova [3] extended the results in [2] and solved the problem of interpolation of scattered data by minimum $L_p$-norms networks for $1 < p < \infty$. The minimum $L_p$-norm network for $1 < p < \infty$ is obtained from the solution to a system of nonlinear equations with coefficients determined by the data. The solution in the case $1 < p < \infty$ is unique. We denote the corresponding minimum $L_p$-norm network by $F_p$.

In this paper we consider the case $p = \infty$ and establish the existence of a solution of the same type as in the case where $1 < p < \infty$. This solution on each edge of the underlying triangulation is a perfect quadratic spline function with at most one knot. We denote this solution by $F_\infty$.

The following question naturally arises. Does the solutions for $1 < p < \infty$ converge to $F_\infty$ as $p \to \infty$? Here we answer positively to this question.

## PRELIMINARIES AND RELATED WORK

Let $n \geq 3$ be an integer and $P_i := (x_i, y_i, z_i)$, $i = 1, \ldots, n$ be different points in $\mathbb{R}^3$. We call this set of points *data*. The data are *scattered*[1] if the projections $V_i := (x_i, y_i)$ onto the plane $Oxy$ are different and non-collinear.

[1] Note that this definition of scattered data slightly misuses the commonly accepted meaning of the term. It allows data with some structure among points $\mathbf{v}_i$. We have opted to do this in order to cover all cases where our presentation and results are valid.

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**Definition 1** *A collection of non-overlapping, non-degenerate triangles in $\mathbb{R}^3$ is a triangulation of the points $V_i,\ i = 1,\dots,n$, if the set of the vertices of the triangles coincides with the set of the points $V_i,\ i=1,\dots,n$.*

Hereafter we assume that a triangulation $T$ of the points $V_i,\ i=1,\dots,n$, is given and fixed. The union of all triangles in $T$ is a polygonal domain which we denote by $D$. In general $D$ is a collection of polygons with holes. The set of the edges of the triangles in $T$ is denoted by $E$. If there is an edge between $V_i$ and $V_j$ in $E$, it will be referred to by $e_{ij}$ or simply by $e$ if no ambiguity arises.

**Definition 2** *A curve network is a collection of real-valued univariate functions $\{f_e\}_{e\in E}$ defined on the edges in $E$.*

With any real-valued bivariate function $F$ defined on $D$ we naturally associate the curve network defined as the restriction of $F$ on the edges in $E$, i. e. for $e = e_{ij} \in E$,

$$\begin{gathered} f_e(t) := F\Big(\big(1-\frac{t}{\|e\|}\big)x_i + \frac{t}{\|e\|}x_j,\ \big(1-\frac{t}{\|e\|}\big)y_i + \frac{t}{\|e\|}y_j\Big), \\ \text{where } 0 \le t \le \|e\| \text{ and } \|e\| = \sqrt{(x_i-x_j)^2+(y_i-y_j)^2}. \end{gathered} \tag{1}$$

Furthermore, according to the context $F$ will denote either a real-valued bivariate function or a curve network defined by (1). For $p$, such that $1 < p \le \infty$, we introduce the following class of *smooth interpolants*

$$\mathscr{F}_p := \{F(x,y) \in C^1(D)\,|\,F(x_i,y_i) = z_i,\ i=1,\dots,n,\ f_e' \in AC,\ f_e''\ \in L_p,\ e\in E\},$$

where $C^1(D)$ is the class of bivariate functions defined in $D$ which possess continuous first order partial derivatives, $AC$ is the class of univariate absolutely continuous functions defined in $[0,\|e\|]$, $L_p$ for $1<p<\infty$ is the class of univariate functions defined in $[0,\|e\|]$ whose p-th power of the absolute value is Lebesgue integrable, and $L_\infty$ is the class of bounded univariate functions defined in $[0,\|e\|]$. The restrictions on $E$ of the functions in $\mathscr{F}_p$ form the corresponding class of so-called *smooth interpolation curve networks*

$$\mathrm{C}_p(E) := \left\{F_{|E} = \{f_e\}_{e\in E}\ |\ F(x,y)\in\mathscr{F}_p,\ e\in E\right\}. \tag{2}$$

The smoothness of the interpolation curve network $F = \{f_e\}_{e\in E} \in \mathrm{C}_p(E)$ geometrically means that at each point $P_i$ there is a *tangent plane* to $F$, where a plane is *tangent* to the curve network at the point $P_i$ if it contains the tangent vectors at $P_i$ of the curves incident to $P_i$. Further, $L_p$-norm are defined in $\mathrm{C}_p(E)$ by

$$\begin{aligned} \|F\|_p &:= \left(\sum_{e\in E}\int_0^{\|e\|}|f_e(t)|^p dt\right)^{1/p}, \quad 1<p<\infty, \\ \|F\|_\infty &:= \max_{e\in E}\ \|f_e\|_\infty. \end{aligned}$$

We denote the networks of the second derivative of $F$ by $F'' := \{f_e''\}_{e\in E}$ and consider the following extremal problem:

$$(\mathbf{P}_p)\qquad \textit{Find } F^*\in\mathrm{C}_p(E)\ \textit{ such that } \|F^{*\prime\prime}\|_p = \inf_{F\in\mathrm{C}_p(E)}\|F''\|_p.$$

Problem $(P_p)$ is a generalization of the classical univariate extremal problem $(\tilde{P}_p)$ for interpolation of data in $\mathbb{R}^2$ by a univariate function with minimal $L_p$-norm of the second derivative. Favard [13] considered problem $(\tilde{P}_p)$ in the case where $p=\infty$ and later Karlin [14] showed that there exists a perfect spline (not necessarily unique) which is a solution to the problem.[2] For $p=2$ Holladay [15] proved that the natural interpolating cubic spline is the unique solution to $(\tilde{P}_2)$. C. de Boor [16] studied in detail and generalized Favard's result [13] for $1<p\le\infty$. We note that Nielson's idea to obtain the minimum norm network is similar to Holladay's proof [15].

For $i=1,\dots,n$, let $m_i$ denote the degree of the vertex $V_i$, i. e. the number of the edges in $E$ incident to $V_i$. Furthermore, let $\{e_{ii_1},\dots,e_{ii_{m_i}}\}$ be the edges incident to $V_i$ listed in clockwise order around $V_i$. The first edge $e_{ii_1}$ is

[2] Favard [13] and Karlin [14] studied the more general problem of minimum $L_p$-norm of the $k$-th derivative, $k\ge 1$.

chosen so that the coefficient $\lambda_{1,i}^{(s)}$ defined below is not zero - this is always possible. A *basic curve network* $B_{is}$ is defined on $E$ for any pair of indices $is$, such that $i = 1, \ldots, n$ and $s = 1, \ldots, m_i - 2$, as follows:

$$B_{is} := \begin{cases} \lambda_{r,i}^{(s)} \left(1 - \frac{t}{\|e_{ii_{s+r-1}}\|}\right) & \text{on } e_{ii_{s+r-1}},\ r = 1,2,3, \\ & \quad 0 \le t \le \|e_{ii_{s+r-1}}\| \\ 0 & \text{on the other edges of } E. \end{cases}$$

The coefficients $\lambda_{r,i}^{(s)}$, $r = 1,2,3$, are uniquely determined to sum to one and to form a zero linear combination of the three unit vectors along the edges $e_{ii_{s+r-1}}$ starting at $V_i$. Note that basic curve networks are associated with points that have at least three edges incident to them. We denote by $N_B$ the set of pairs of indices $is$ for which a basic curve network is defined, i. e., $N_B := \{is \mid m_i \ge 3,\ i = 1, \ldots, n,\ s = 1, \ldots, m_i - 2\}$.

With each basic curve network $B_{is}$ for $is \in N_B$ we associate a number $d_{is}$ defined by

$$d_{is} = \frac{\lambda_{1,i}^{(s)}}{\|e_{ii_s}\|}(z_{i_s} - z_i) + \frac{\lambda_{2,i}^{(s)}}{\|e_{ii_{s+1}}\|}(z_{i_{s+1}} - z_i) + \frac{\lambda_{3,i}^{(s)}}{\|e_{ii_{s+2}}\|}(z_{i_{s+2}} - z_i),$$

which reflects the position of the data in the supporting set of $B_{is}$.

In [3] a full characterization of the solution $F_p$ to the extremal problem $(P_p)$ for $1 < p < \infty$ was provided. Finding of $F_p$ reduces to the unique solution of a system of equations. Further, for simplicity we use the notation $(x)_{\pm}^{r} := |x|^r \operatorname{sign}(x)$, $r \in \mathbb{R}, x \in \mathbb{R}$. The following two theorems were proved in [3].

**Theorem 1** *The extremal problem $(P_p)$ has always a unique solution for $1 < p < \infty$.*

**Theorem 2** *Curve network $F_p \in C_p(E)$ solves problem $(P_p)$ for $1 < p < \infty$ if and only if $F_p'' = \left(\sum_{is \in N_B} \alpha_{is} B_{is}\right)_{\pm}^{q-1}$. The coefficients $\alpha_{is}$ are the unique solution to the following system of equations*

$$\int_E \left(\sum_{is \in N_B} \alpha_{is} B_{is}\right)_{\pm}^{q-1} B_{kl}\, dt = d_{kl},\ kl \in N_B. \tag{3}$$

## MAIN RESULTS

In Theorem 1 we proved that problem $(P_p)$ for $1 < p < \infty$ has always a unique solution, and in Theorem 2 we characterized this solution using Lagrange multipliers. This approach can not be applied to the case where $p = \infty$. By a different approach used by de Boor [16] in the univariate case, in the next theorem we establish the existence of a solution to $(P_\infty)$ and determine its form.

**Theorem 3** *Problem $(P_\infty)$ has a solution $F_\infty \in C_\infty(E)$ such that*

$$F_\infty'' = c \left(\sum_{is \in N_B} \alpha_{is} B_{is}\right)_{\pm}^{0} \quad a.\ e.\ on\ E$$

*provided $(\sum_{is \in N_B} \alpha_{is} B_{is})_{|e} \not\equiv 0$ for every $e \in E$. Coefficients $\alpha_{is}$ are real numbers and $c > 0$.*

**Remark 1** *Coefficients $\alpha_{is}$ are determined up to a constant factor.*

**Corollary 1** *Coefficients $c$ and $\alpha_{is}$, $is \in N_B$, are a solution to the system*

$$\int_E c \left(\sum_{is \in N_B} \alpha_{is} B_{is}\right)_{\pm}^{0} B_{kl} = d_{kl},\ kl \in N_B. \tag{4}$$

It is natural to ask the following question. Does the solutions $F_p$ for $1 < p < \infty$ converge as $p \to \infty$ and what is the limit? The next theorem gives answer to this question.

**Theorem 4** *The minimum $L_p$-norm networks $F_p$ converge to the minimum $L_\infty$-norm network $F_\infty$ as $p \to \infty$.*

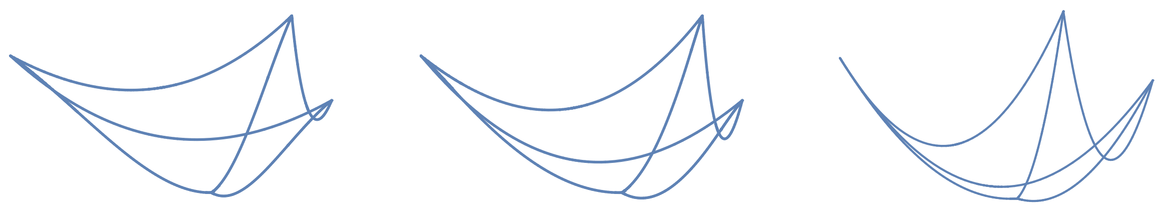
## NUMERICAL EXPERIMENTS

To find the minimum $L_p$-norm networks for $1 < p < \infty$ we have to solve system (3) which is nonlinear except in the case where $p = 2$ when it is linear. We have implemented a Newton type algorithm [17] to solve this type of systems. We use Mathematica package to visualize the extremal curve networks $F_p$. Here we present an example from our experimental work.

**Example 1** *We consider data obtained from a regular triangular pyramid. We have* $n = 4$, $V_1 = (-1/2, -\sqrt{3}/6)$, $V_2 = (1/2, -\sqrt{3}/6)$, $V_3 = (0, \sqrt{3}/3)$, $V_4 = (0,0)$, *and* $z_i = 0$, $i = 1,2,3$, $z_4 = -1/2$. *The set of indices defining the edges of the corresponding triangulation* $T$ *is* $N_B = \{12, 23, 31, 41, 42, 43\}$. *We have* $m_i = 3$ *for* $i = 1, \ldots, 4$ *and four basic curve networks* $B_{is}$, $i = 1, \ldots, 4$, $s = 1$, *are defined. The exact solution for* $p = \infty$ *can be found directly by solving system (4). It is*

$$f_{12}(t) = f_{23}(t) = f_{31}(t) = 3(t^2 - t)/2,\ 0 \le t \le 1; \quad f_{j4}(t) = 3t^2/2 - \sqrt{3}t,\ 0 \le t \le \sqrt{3}/3, \quad j = 1,2,3.$$

*In Fig. 1 the minimum* $L_p$*-norms network* $F_p$*, and the corresponding* $L_p$*-norms of the second derivatives* $\|F_p''\|_p$ *for* $p = 2, 6,$ and $\infty$*, are shown.*



$p = 2,\ \|F_2''\|_2 = 4.72119$ $\quad$ $p = 6,\ \|F_6''\|_6 = 3.40846$ $\quad$ $p = \infty,\ \|F_\infty''\|_\infty = 3.$

**FIGURE 1.** Example 1: The minimum $L_p$-norm networks $F_p$, and the corresponding $L_p$-norms $\|F_p''\|_p$ for $p = 2, 6,$ and $\infty$.

## REFERENCES


1. G. Nielson, "A method for interpolating scattered data based upon a minimum norm network," Math. Comput. **40**, 253–271 (1983).
2. L. Andersson, T. Elfving, G. Iliev, and K. Vlachkova, "Interpolation of convex scattered data in $\mathbb{R}^3$ based upon an edge convex minimum norm network," J. of Approx. Theory **80**, 299 – 320 (1995).
3. K. Vlachkova, "Interpolation of scattered data in $\mathbb{R}^3$ using minimum $l_p$-norm network, $1 < p < \infty$," J. Math. Anal. Appl. **482**, 123824 (2020).
4. T. Foley and H. Hagen, "Advances in scattered data interpolation," Surv. Math. Ind. **4**, 71–84 (1994).
5. R. Franke and G. Nielson, "Scattered data interpolation and applications: a tutorial and survey," in *Geometric Modeling*, edited by H. Hagen and D. Roller (Springer, Berlin, 1991) pp. 131–160.
6. S. Lodha and K. Franke, "Scattered data techniques for surfaces," in *Proceedings of Dagstuhl Conference on Scientific Visualization* (IEEE Computer Society Press, Washington, 1997) pp. 182–222.
7. S. Mann, C. Loop, M. Lounsbery, D. Meyers, J. Painter, T. DeRose, and K. Sloan, "A survey of parametric scattered data fitting using triangular interpolants," in *Curve and Surface Design*, edited by H. Hagen (SIAM, Philadelphia, 1992) pp. 145–172.
8. T. K. Dey, *Curve and Surface Reconstruction: Algorithms with Mathematical Analysis*, Cambridge Monographs on Applied and Computational Mathematics (Cambridge University Press, 2006).
9. I. Amidror, "Scattered data interpolation methods for electronic imaging systems: a survey," J. of Electron. Imaging **11**, 157–176 (2002).
10. K. Anjyo, J. Lewis, and F. Pighin, "Scattered data interpolation for computer graphics," in *Proceedings SIGGRAPH'14 ACM Courses Article No. 27* (2014) pp. 1–69.
11. F. Cazals and J. Giesen, "Delaunay triangulation based surface reconstruction," in *Effective computational geometry for curves and surfaces*, edited by J.-D. Boissonat and M. Teillaud (Springer, Berlin Heidelberg, 2006) pp. 231–276.
12. F. Dell'Accio and F. D. Tommaso, "Scattered data interpolation by Shepard's like methods: classical results and recent advances," Dolomites Res. Notes Approx. **9**, 32 – 44 (2016).
13. J. Favard, "Sur l'interpolation," J. Math. Pures Appl. **19**, 281–306 (1976).
14. S. Karlin, "Interpolation properties of generalized perfect splines and the solutions of certain extremal problems," I. Trans. Amer. Math. Soc. **106**, 25–66 (1975).
15. J. Holladay, "A smoothest curve approximation," Math. Tables Other Aids Comput. **11**, 233–243 (1957).
16. C. de Boor, "On "best" interpolation," J. of Approx. Theory **16**, 28–42 (1976).
17. K. Vlachkova, "A Newton-type algorithm for solving an extremal constrained interpolaion problem," Numer. Linear Algebra Appl. **7**, 133–146 (2000).