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Scattered Data Interpolation in $\mathbb{R}^3$ by Smooth Curve Networks

Krassimira Vlachkova

Given scattered data in $\mathbb{R}^3$ and an integer $k \geq 1$ we consider the problem of finding necessary and sufficient conditions for k-times smoothness of interpolation curve networks. Interpolation curve networks are defined on the edges of a triangulation associated with the projections of the data onto a fixed plane and play an essential role in construction of smooth interpolation surfaces in $\mathbb{R}^3$. In previous work this problem has been solved for $k = 1$ by the aid of a geometric criterion which does not generalize for $k > 1$. In this paper we apply a different approach and obtain necessary and sufficient conditions for k-times smoothness of interpolation curve networks.

1. Introduction

The problem of interpolating scattered data in $\mathbb{R}^3$ by smooth surfaces has a wide range of applications in both theory and practice and received considerable attention in the last decades, see, e.g., the surveys [2], [3], [4]. The problem can be formulated as follows: Given a set of points $(x_i, y_i, z_i) \in \mathbb{R}^3$, $i = 1, \ldots, n$, find a bivariate function F possessing continuous partial derivatives up to a given order and such that $F(x_i, y_i) = z_i$. One of the possible approaches to interpolating scattered data is due to Nielson [5]. It consists of the following three steps:

Step 1. Construct a triangulation T of the points $V_i = (x_i, y_i)$, $i = 1, \ldots, n$.

Step 2. Compute a smooth *curve network* interpolant defined on T.

Step 3. Compute a smooth interpolant using an appropriate *blending method*.

Let k be a positive integer. In this paper we concentrate on Step 2 of the above approach. We consider the problem of finding necessary and sufficient conditions for k-times smoothness of an interpolation curve network. Such a characterization is valuable, since it precisely determines the class of k-times smooth networks interpolating given data. Subsequently, the structure of this class can be studied (e.g., by constructing an appropriate basis) to obtain new theoretical results and efficient numerical methods for finding k-times smooth interpolation networks, see [1] and [6] for the case $k = 1$. In [1], among others, the problem has been solved for $k = 1$ by the aid of a geometric criterion which does not generalize for $k > 1$. Here we propose a different approach and solve the problem for any $k \geq 1$.

The paper is organized as follows: In Section 2 we introduce notation and formulate the problem. In Section 3 we obtain the necessary and sufficient conditions for k-times smoothness. The main results are summarized in Theorems 3 and 4.

2. Notation and Problem Formulation

Let $n \geq 3$ be an integer and $P_i := (x_i, y_i, z_i),\ i = 1, \ldots, n$, be different points in $\mathbb{R}^3$. We call this set of points *data*. The data are *scattered* if the projections $V_i := (x_i, y_i)$ onto the plane $z = 0$ are different and non-colinear.

Next, a collection of non-overlapping, non-degenerate triangles in $\mathbb{R}^3$ is a *triangulation* of the points $V_i,\ i = 1, \ldots, n$, if the set of the vertices of the triangles coincides with the set of the points $V_i,\ i = 1, \ldots, n$. Let T be a given triangulation of the points $V_i,\ i = 1, \ldots, n$. The union of triangles in T is a polygonal domain which we denote by D. In general D is a collection of polygons with holes. The set of the edges of the triangles in T is denoted by E. If there is an edge between V_i and V_j in E, it will be referred to by e_{ij} or simply by e if no ambiguity arises. Similarly, $\mathbf{e_{ij}}$ or simply $\mathbf{e}$ will denote the vector corresponding to e_{ij} starting at V_i.

Definition 1. *A curve network is a collection of real-valued univariate functions $\{f_e\}_{e \in E}$ defined on the edges in E.*

With any real-valued bivariate function F defined on D we naturally associate the curve network defined as the restriction of F on the edges in E, i.e., for $e = e_{ij} \in E$,

$$\begin{aligned} &f_e(t) := F\left(\left(1 - \tfrac{t}{\|e\|}\right)x_i + \tfrac{t}{\|e\|}x_j,\ \left(1 - \tfrac{t}{\|e\|}\right)y_i + \tfrac{t}{\|e\|}y_j\right), \\ &\text{where } 0 \leq t \leq \|e\|, \text{ and } \|e\| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}. \end{aligned} \tag{1}$$

Furthermore, according to the context, F will denote either a real-valued bivariate function or a curve network defined by (1). For every integer $k \geq 1$ let $\mathcal{F}^k$ be the class of k-times smooth interpolants defined by:

$$\mathcal{F}^k := \left\{F(x,y) : F(x_i, y_i) = z_i,\ i = 1, \ldots, n,\ \frac{\partial^k F}{\partial x^r \partial y^{k-r}} \in C(D),\ r = 0, \ldots, k\right\},$$

where $C(D)$ is the class of bivariate continuous functions defined on D. We define the class of *k-times smooth interpolation curve networks* by

$$\mathcal{C}^k(E) := \left\{F = \{f_e\}_{e \in E} : F(x,y) \in \mathcal{F}^k\right\} \ .$$

In the next section we obtain necessary and sufficient conditions for an interpolation curve network $F = \{f_e\}_{e\in E}$ to be in the class $\mathcal{C}^k(E)$.

3. Necessary and Sufficient Conditions for k-times Smoothness

Let α_e denote the oriented angle between $\mathbf{e}$ and the positive direction of the x-axis. Then we have

$$f_e(t) = F(x,y)_{|e} = F(x_i + t\cos\alpha_e, y_i + t\sin\alpha_e), \qquad 0 \le t \le \|e\|. \tag{2}$$

We obtain our conditions inductively with respect to k. Let us consider the case $k = 1$ first. Taking derivatives in (2) we obtain

$$\begin{aligned} f_e'(t) =& \frac{\partial F}{\partial x}(x_i + t\cos\alpha_e, y_i + t\sin\alpha_e)\cos\alpha_e \\ &+ \frac{\partial F}{\partial y}(x_i + t\cos\alpha_e, y_i + t\sin\alpha_e)\sin\alpha_e. \end{aligned} \tag{3}$$

Let m_i denote the degree of the vertex V_i, i.e., the number of the edges incident to V_i. Let $m_i \ge 3$ and e_{il}, e_{im} and e_{ir} be three arbitrary edges incident to V_i. From (3), with $t = 0$, we obtain the following system of linear equations in the unknowns $\partial F/\partial x(V_i)$ and $\partial F/\partial y(V_i)$:

$$\left| \begin{array}{lcl} f_{il}'(0) &=& \dfrac{\partial F(V_i)}{\partial x}\cos\alpha_{il} + \dfrac{\partial F(V_i)}{\partial y}\sin\alpha_{il} \\ f_{im}'(0) &=& \dfrac{\partial F(V_i)}{\partial x}\cos\alpha_{im} + \dfrac{\partial F(V_i)}{\partial y}\sin\alpha_{im} \\ f_{ir}'(0) &=& \dfrac{\partial F(V_i)}{\partial x}\cos\alpha_{ir} + \dfrac{\partial F(V_i)}{\partial y}\sin\alpha_{ir}. \end{array} \right. \tag{4}$$

Since $F \in C^1(D)$ then $\partial F/\partial x$ and $\partial F/\partial y$ are continuous in D. In particular, they are continuous at the point V_i. Therefore the system (4) has a solution. Now we shall apply Rouchet theorem which states that the necessary and sufficient condition for an $m \times n$ system of linear equations $Ax = b$ to have a solution is the rank $r(A)$ of the coefficient matrix to be equal to the rank of the augmented matrix. Moreover, the number of the solutions of the system depends on $n - r(A)$ parameters. In our case the rank of the coefficient matrix of (4) is equal to 2, since at least two of the edges $e_{il},\ e_{im},\ e_{ir}$ are not co-linear and therefore there exists a minor of rank 2. Thus, the necessary and sufficient condition for (4) to have a unique solution is the following determinant to be zero:

$$\begin{vmatrix} f_{il}'(0) & \cos\alpha_{il} & \sin\alpha_{il} \\ f_{im}'(0) & \cos\alpha_{im} & \sin\alpha_{im} \\ f_{ir}'(0) & \cos\alpha_{ir} & \sin\alpha_{ir} \end{vmatrix} = 0. \tag{5}$$

In the case $m_i = 2$, i.e., V_i is a boundary point of the domain D, the system (4) consists of two equations only:

$$\left| \begin{array}{lcl} f'_{il}(0) & = & \dfrac{\partial F(V_i)}{\partial x} \cos\alpha_{il} + \dfrac{\partial F}{\partial y}(V_i) \sin\alpha_{il} \\ f'_{im}(0) & = & \dfrac{\partial F}{\partial x}(V_i) \cos\alpha_{im} + \dfrac{\partial F}{\partial y}(V_i) \sin\alpha_{im}. \end{array} \right. \tag{6}$$

Since e_{il} and e_{im} are edges of the triangulation then they are not co-linear. Therefore the determinant of (6), which is equal to $\sin(\alpha_{im} - \alpha_{il})$, is not zero. Thus (6) has always a unique solution.

Let $N_i := \{i_1, \ldots, i_{m_i}\}$ be the set of indices of the vertices adjacent to V_i in T listed in a clockwise order around V_i starting from an arbitrarily chosen index i_1. We can formulate the following theorem and corollary.

Theorem 1. *A curve network $F = \{f_e\}_{e\in E}$ is smooth, i.e., $F \in \mathcal{C}^1(E)$, if and only if for every vertex V_i of degree $m_i \geq 3$ and for every three different indices $k, l, m \in N_i$, the condition* (5) *holds.*

Corollary 1. *A curve network $F = \{f_e\}_{e\in E}$ is smooth if and only if for every vertex V_i of degree $m_i \geq 3$ and $r = 1, \ldots, m_i - 2$, it holds*

$$\begin{vmatrix} f'_{ii_r}(0) & \cos\alpha_{ii_r} & \sin\alpha_{ii_r} \\ f'_{ii_{r+1}}(0) & \cos\alpha_{ii_{r+1}} & \sin\alpha_{ii_{r+1}} \\ f'_{ii_{r+2}}(0) & \cos\alpha_{ii_{r+2}} & \sin\alpha_{ii_{r+2}} \end{vmatrix} = 0. \tag{7}$$

Conditions (7) *for $i = 1, \ldots, n,\ m_i \geq 3$ and $r = 1, \ldots, m_i - 2$ are independent.*

The graphs of the functions defined by (1) can be considered as parametric curves $(x = x_e(t),\ y = y_e(t),\ z = z_e(t)$ in $\mathbb{R}^3$. Then the vector $\mathbf{t_e} := (\cos\alpha_e,\ \sin\alpha_e,\ f'_e(0)) \in \mathbb{R}^3$ is the tangent vector to f_e at the point V_i. Therefore (7) is a necessary and sufficient condition for the tangent vectors $\mathbf{t_{ii_r}}\ \mathbf{t_{ii_{r+1}}}\ \mathbf{t_{ii_{r+2}}}$ to be linearly dependent, i.e., co-planar.

Geometrically the smoothness of $\{f_e\}_{e\in E}$ means that at every point V_i, $i = 1, \ldots, n$, the curves have a common tangent plane. In [1] this geometric criterion is used to obtain the conditions (7) for $k = 1$. But it is not clear how this geometric approach can be generalized for $k > 1$. The approach used here allows us to obtain necessary and sufficient conditions for arbitrary $k > 1$.

Definition 2. *The vertex V_i is singular if there are two co-linear edges starting from V_i.*

Definition 3. *The data are nonsingular if all vertices $V_i,\ i = 1, \ldots, n$, are nonsingular.*

Theorem 2. *Let the data be nonsingular. A curve network $F = \{f_e\}_{e\in E}$ is k-times smooth, i.e., $F \in \mathcal{C}^k(E)$, if and only if $F \in \mathcal{C}^{k-1}(E)$ and for every vertex V_i of degree $m_i \geq k+2$ and for every $k+2$ different indices $j_1, \ldots, j_{k+2} \in N_i$, it holds*

$$\begin{vmatrix} f^{(k)}_{ij_1}(0) & \cos^k \alpha_{ij_1} & \ldots & \sin^k \alpha_{ij_1} \\ \cdots & \cdots & \cdots & \cdots \\ f^{(k)}_{ij_{k+2}}(0) & \cos^k \alpha_{ij_{k+2}} & \ldots & \sin^k \alpha_{ij_{k+2}} \end{vmatrix} = 0. \tag{8}$$

Proof. Consider an interpolation curve network corresponding to $F \in \mathcal{F}^k$. Taking k-th derivatives in (2) we obtain

$$f^{(k)}_e(t) = \sum_{r=0}^{k} \binom{k}{r} \frac{\partial^k F(x_i + t\cos\alpha_e, y_i + t\sin\alpha_e)}{\partial x^r \partial y^{k-r}} \cos^{k-r}\alpha_e \sin^r \alpha_e. \tag{9}$$

Let $m \leq m_i$ and $j_1, \ldots, j_m \in N_i$ be m different indices. We consider the following system of m linear equations in the $k+1$ unknowns $\partial^k F/\partial x^r \partial y^{k-r}(V_i)$, $r = 0, \ldots, k$, which is obtained from (9) for $t = 0$ and $j = j_1, \ldots, j_m$:

$$f^{(k)}_{ij}(0) = \sum_{r=0}^{k} \binom{k}{r} \frac{\partial^k F(V_i)}{\partial x^r \partial y^{k-r}} \cos^{k-r}\alpha_{ij} \sin^r \alpha_{ij}. \tag{10}$$

Next, we study the rank of the system (10) by applying Rouchet theorem. Suppose first that $m = k+1$. Then (10) has exactly $k+1$ unknowns and $k+1$ equations. The determinant D of its coefficient matrix can be computed as follows. First, assume that $\alpha_{ij_r} \neq 0, \pi$ for $r = 1, \ldots, k+1$. Then we have

$$\begin{aligned} D &= \prod_{r=0}^{k} \binom{k}{r} \begin{vmatrix} \cos^k \alpha_{ij_1} & \cos^{k-1}\alpha_{ij_1} \sin\alpha_{ij_1} & \ldots & \sin^k \alpha_{ij_1} \\ \cdots & \cdots & \cdots & \cdots \\ \cos^k \alpha_{ij_{k+1}} & \cos^{k-1}\alpha_{ij_{k+1}} \sin\alpha_{ij_{k+1}} & \ldots & \sin^k \alpha_{ij_{k+1}} \end{vmatrix} \\ &= \prod_{r=0}^{k} \binom{k}{r} \prod_{p=1}^{k+1} \sin^k \alpha_{ij_p} V(\cot\alpha_{ij_1}, \ldots, \cot\alpha_{ij_{k+1}}) \\ &= \prod_{r=0}^{k} \binom{k}{r} \prod_{p=1}^{k+1} \sin^k \alpha_{ij_p} \prod_{r>s} (\cot\alpha_{ij_r} - \cot\alpha_{ij_s}) \\ &= (-1)^{k(k+1)/2} \prod_{r=0}^{k} \binom{k}{r} \prod_{r>s} \sin(\alpha_{ij_r} - \alpha_{ij_s}). \end{aligned} \tag{11}$$

We used the Vandermonde determinant $V(x_1, \ldots, x_n) = \prod_{r>s}(x_r - x_s)$. The case where one of the angles $\alpha_{ij_r} = 0$ (or π), is treated in the same way obtaining

$$D = \pm \prod_{r=0}^{k} \binom{k}{r} \prod_{r>s} \sin(\alpha_{ij_r} - a_{ij_s}).$$

Since the data are nonsingular then $D \neq 0$ and therefore the rank of the system (10) equals $k+1$. Let $m_i \geq k+2$ and $m = k+2$. Since $F \in \mathcal{F}^k$ then its partial derivatives of order k are continuous and therefore the system (10) has a solution. Since the rank of (10) equals $k+1$ then according to Rouchet theorem, a necessary and sufficient condition for (10) to have a unique solution is the condition (8) to hold.

In the case where $m_i < k+2$, the system (10) has $k+1$ unknowns and at most $k+1$ equations. According to Rouchet theorem there always exists a non-zero solution. $\square$

Since the conditions in (8) are not irrelevant then the next corollary applies.

Corollary 2. *Let the data be nonsingular. A curve network $F = \{f_e\}_{e\in E}$ is k-times smooth if and only if F is $(k-1)$-times smooth and for every vertex V_i of degree $m_i \geq k+2$ and $r = 1, \dots, m_i - k - 1$, it holds*

$$\begin{vmatrix} f_{ii_r}^{(k)}(0) & \cos^k \alpha_{ii_r} & \dots & \sin^k \alpha_{ii_r} \\ \vdots & & & \\ f_{ii_{r+k+1}}^{(k)}(0) & \cos^k \alpha_{ii_{r+k+1}} & \dots & \sin^k \alpha_{ii_{r+k+1}} \end{vmatrix} = 0. \tag{12}$$

Conditions (12) *for $i = 1, \dots, n,\ m_i \geq k+2$ and $r = 1, \dots, m_i - k - 1$ are independent.*

We summarize the above in the next theorem, which is our main result.

Theorem 3. *Let the data be nonsingular. The interpolation curve network $F = \{f_e\}_{e\in E}$ is k-times smooth if and only if for every $i = 1, \dots, n$*

$$\begin{aligned} &\sum_{p=1}^{l+2} f_{r+p}^{(l)}(0) A_p^r = 0, \quad \text{for} \\ &l = 1, \dots, \min\{k, m_i - 2\},\ r = 1, \dots, m_i - l - 2, \text{ and where} \\ &A_p^r = (-1)^{p+1} \prod_{m>s;\, m,s\neq p}^{r+k+1} \sin(\alpha_{ii_m} - \alpha_{ii_s}). \end{aligned} \tag{13}$$

It remains to consider the case where V_i is a singular vertex. Let there exist p couples co-linear edges, say e_{ij_r} and $e_{ij_{r+p}}$ for $r = 1, \dots, p$, where $2p \leq m_i$. Applying Rouchet theorem to the system (10) we obtain

Theorem 4. *The curve network $F = \{f_e\}_{e\in E}$ is k-times smooth at V_i if and only if*

(i) $f_{ij_r}^{(l)}(0) = (-1)^l f_{ij_{r+p}}^{(l)}(0)$, *for $l = 1, \dots, k$, $r = 1, \dots, p$.*

(ii) *Conditions* (13) *hold for the set of indices $j_1, \dots j_p, j_{2p+1}, \dots, j_{m_i}$.*

Remark. The results in this paper can be easily extended from a triangulation to an arbitrary mesh on the vertices V_i, $i = 1, \dots, n$, e.g. rectangular, hexagonal or irregular mesh, due to the fact that any such mesh can be obtained from a certain triangulation by removing some of its edges.

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Krassimira Vlachkova
Department of Mathematics
University of Sofia
Blvd. James Bourchier 5
1164 Sofia
BULGARIA
E-mail: `krassivl@bas.bg`