Einstein-Weyl structures on certain compact conformal manifolds

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Abstract

We prove that every Einstein-Weyl structure with vanishing conformal Cotton-York tensor on a compact conformal manifold is closed and consequently, the standard metric is either Einstein or locally isometric to $R \times S^n$. As a consequence, we get that the standard metric of an Einstein-Weyl structure with respect to a locally conformal flat structure on a compact manifold is either of constant sectional curvatures or locally isometric to $R \times S^n$.

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1 Introduction

Manifolds with conformal structure $[g]$ and torsion-free affine connection D, such that parallel translation induces conformal transformations, are called Weyl manifolds. The Weyl structure D on a Weyl manifold is described by a pair (g, θ) consisting of a metric $g \in [g]$ and a 1-form θ such that

$$Dg = \theta \otimes g \tag{1.1}$$

If, furthermore the trace-free symmetric part of the Ricci curvature Ric^D of D vanishes, the Weyl structure is said to be Einstein-Weyl. Many examples and general results have been obtained (see [4, 8, 9, 10, 11] and references therein; for Einstein-Weyl structures on complex manifolds see [7, 5]).

The metric $g \in [g]$ is said to be standard if the 1-form θ is co-closed with respect to g. If the manifold is compact then Gauduchon [3] shows that, up to homothety, there is a unique choice of the standard metric in the conformal class [g].

A Weyl structure is said to be closed if $d\theta = 0$.

An invariant of a given Weyl structure is the conformal Cotton-York tensor.

In dimension 4, Weyl structures with zero conformal Cotton-York tensor play an important role in General relativity [6, 1]. The vanishing of the conformal Cotton-York tensor implies the vanishing of the Eastwood-Dighton tensor and if the self-dual part W^+ and the anti-self-dual part W^- of the Weyl curvature have non-degenerate spectrum simultaneously, then the Weyl structure is closed and locally conformally equivalent to a metric with harmonic Weyl curvature [6, 1]. If moreover, the Bach tensor is also zero, then the conformal structure contains an Einstein metric [1, 6]. If W^+ or W^- has degenerate spectrum however, the situation is more complicated. In the compact case, Gauduchon [4] classified, up to a conformal isometry, self-dual Einstein-Weyl structures and Pedersen and Swann described in [8] Einstein-Weyl 4-manifolds with vanishing Bach tensor.

Examples, in all dimensions, of Weyl structures with vanishing conformal Cotton-York tensor are the Weyl structures admitting a metric with harmonic Weyl tensor as well as Weyl structures with an Einstein metric in the same conformal class. General results and examples of Riemannian manifolds with harmonic Weyl tensor and Einstein manifolds are presented in [2].

In this note we study Einstein-Weyl structures with zero conformal Cotton-York tensor in all dimensions grater than two. The aim of the paper is to prove the following

Theorem 1.1 *Any Einstein-Weyl structure with vanishing conformal Cotton-York tensor on a compact conformal n-manifold ($n \geq 3$) is closed.*

The closed Einstein-Weyl structures on a compact manifold are classified by Gauduchon in [4]. Applying Theorem 3 in [4], we may formulate Theorem 1.1 as follows

Theorem 1.1' *Let g be the standard metric of an Einstein-Weyl structure D with vanishing conformal Cotton-York tensor on a compact conformal manifold of dimension n, ($n \geq 3$). Then:*

i) either D coincides with the Levi-Civita connection of g,

or

ii) D is flat, the metric g is locally isometric (up to homothety) to the product metric of $R \times N^{n-1}$, where N is an Einstein space, *and the 1-form θ is parallel with respect to the Levi-Civita connection of g.*

As a consequence, we get

Corollary 1.2 *Let g be the standard metric of an Einstein-Weyl structure D on a compact locally conformally flat conformal manifold of dimension n ($n \geq 3$). Then:*

i) either D coincides with the Levi-Civita connection of g and g is of constant sectional curvature, or

ii) D is flat, the metric g is locally isometric (up to homothety) to the standard product metric of $R \times S^{n-1}$ and the 1-form θ is parallel with respect to the Levi-Civita connection of g.

Remark 1.1 In four dimensions an Einstein-Weyl manifold with vanishing conformal Cotton-York tensor is easily seen to be Bach-flat and Theorem 1.1 follows from the result of Pedersen and Swann (Corollary 3.2 in [8]).

Remark 1.2. In dimensions four it is clear that every locally conformally flat conformal manifold is self dual conformal manifold and Corollary 1.2 follows from the results of Gauduchon (Theorem 4 and Theorem 3 in [4]).

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2 Preliminaries.

Let $(\mathbf{M}, [g])$ be an n-dimensional ($n \geq 3$) conformal manifold with Weyl structure D. We denote by Ric^D the Ricci curvature of D. The conformal scalar curvature $Scal^D$ of D is defined by $Scal^D = tr_g(Ric^D)$, where tr_g is the trace with respect to a metric $g \in [g]$. The normalized Ricci tensor h^D of D is given by the expression

$$h^D = \frac{1}{n-2}\left(Ric^D - \frac{Scal^D}{2(n-1)}g\right).$$

The conformal Cotton-York tensor C^D of D is defined by

$$C^D(X,Y,Z) = (D_Y h^D)(X,Z) - (D_X h^D)(Y,Z), \quad X,Y,Z \in T_p\mathbf{M}, p \in \mathbf{M},$$

where $T_p\mathbf{M}$ is the tangential space at a point $p \in \mathbf{M}$.

It is clear from the definition that C^D is a conformal invariant.

Let D^g be the Levi-Civita connection of a metric $g \in [g]$. Then the condition (1.1) is equivalent to

$$D_X Y = D^g_X Y - \frac{1}{2}\left(\theta(X)Y + \theta(Y)X - g(X,Y)T\right) \tag{2.2}$$

for any smooth vector fields X, Y on $\mathbf{M}$, where T denotes the dual vector field corresponding to the 1-form θ with respect to g.

Let C^g be the Cotton-York tensor of the metric g. The second Bianchi identity implies the formula $\delta W = C^g$, where W denotes the Weyl curvature and δW is the co-differential of W with respect to the metric g. So, the Riemannian manifolds with zero Cotton-York tensor are precisely the Riemannian manifolds with harmonic Weyl tensor.

As a consequence of (2.2), the connection between the conformal Cotton-York tensor C^D of D and the Cotton-York tensor C^g of g is given by [4]

$$C^D(X,Y,Z) = C^g(X,Y,Z) + \theta(W(X,Y)Z), \quad X,Y,Z \in T_p\mathbf{M}, p \in \mathbf{M}, \tag{2.3}$$

where W is the Weyl conformal tensor of g.

Now, let D be an Einstein-Weyl connection, i.e. the symmetric part of the Ricci tensor Ric^D of D is proportional to the metric g at each point of $\mathbf{M}$. In terms of the Ricci tensor Ric^g of the metric g the Einstein-Weyl equations are [9, 8, 4]

$$Ric^g = \frac{2-n}{4}\left(2Sym(D^g\theta) + \theta \otimes \theta\right) + \left(\frac{1}{n}Scal^D + \frac{n-2}{4}\|\theta\|^2 + \frac{1}{2}\delta\theta\right)g, \tag{2.4}$$

where $Sym(D^g\theta)$ denotes the symmetric part of $D^g\theta$, $\|.\|^2$ is the usual tensor norm and $\delta\theta$ is the co-differential of θ with respect to g.

3 Proof of Theorem 1.1

Let D be an Einstein-Weyl structure on a compact conformal manifold $(\mathbf{M},[g])$. From now on, the metric $g \in [g]$ will stand for the standard metric of the conformal class $[g]$. We shall denote the corresponding Levi-Civita connection by ∇. For the standard metric on a compact manifold the vector field T dual to θ is Killing [11] and we may write the Einstein-Weyl equations (2.4) in the form [9, 8, 4]

$$Ric^g = \frac{1}{n} Scal^D g + \frac{n-2}{4}\left(\|\theta\|^2 g - \theta \otimes \theta\right). \tag{3.5}$$

For the conformal scalar curvature, we obtain from (3.5) the formula [9, 8, 4]

$$Scal^g = Scal^D + \frac{(n-1)(n-2)}{4}\|\theta\|^2. \tag{3.6}$$

The second Bianchi identity for ∇ implies [4]

$$4Scal^D + n(n-4)\|\theta\|^2 = const, \tag{3.7}$$

For the normalized Ricci tensor h^g of g, we calculate using (3.5) and (3.6)

$$h^g = \frac{4Scal^D + n(n-1)\|\theta\|^2}{8n(n-1)} g - \frac{1}{4}\theta \otimes \theta$$

Taking into account (3.7) we rewrite the latter equality in the form

$$h^g = const.g + \frac{3}{8(n-1)}\|\theta\|^2 g - \frac{1}{4}\theta \otimes \theta. \tag{3.8}$$

Let the conformal Cotton-York tensor C^D of D be zero. Then (2.3) implies

$$C^D(X,Y,T) = C^g(X,Y,T) = 0, \quad X,Y \in T_p\mathbf{M}, p \in \mathbf{M}. \tag{3.9}$$

For the Cotton-York tensor C^g of g, we obtain using (3.8)

$$C^g = -\frac{1}{4} d^\nabla \left(\frac{3}{2(n-1)}\|\theta\|^2 g - \theta \otimes \theta)\right), \tag{3.10}$$

where d^∇ is a differential operator acting on a symmetric $(0,2)$ tensor S by

$$d^\nabla S(X,Y,Z) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z), \quad X,Y,Z \in T_p\mathbf{M}, p \in \mathbf{M}.$$

Since the vector field T is Killing, we get from (3.10) that

$$C^g(X,Y,Z) = -\frac{3}{2(n-1)} d^\nabla(\|\theta\|^2 g)(X,Y,Z) + 2\theta(Z)(\nabla_X\theta)Y +$$
$$+\theta(Y)(\nabla_X\theta)Z - \theta(X)(\nabla_Y\theta)Z, \quad X,Y,Z \in T_p\mathbf{M}, p \in \mathbf{M}.$$

The latter equality together with (3.9) implies

$$2\|\theta\|^2(\nabla_X\theta)Y = \frac{3}{2(n-1)} d^\nabla(\|\theta\|^2 g)(X,Y,T) \tag{3.11}$$
$$-\theta(Y)(\nabla_X\theta)T + \theta(X)(\nabla_Y\theta)T, \quad X,Y \in T_p\mathbf{M}, p \in \mathbf{M}.$$

We obtain consequently from (3.11)

$$\|\theta\|^2\nabla\theta = \frac{4-n}{4(n-1)}\left(d(\|\theta\|^2\theta) - \|\theta\|^2 d\theta\right), \tag{3.12}$$

$$d(\|\theta\|^2)^2 = \frac{n-4}{3(n-1)(n-2)}T(\|\theta\|^2)\theta. \tag{3.13}$$

Substituting (3.13) into (3.12), we get

$$\|\theta\|^4\nabla\theta = 0 \tag{3.14}$$

The equality (3.14) implies

$$\nabla\theta = 0.$$

Thus, we have proved Theorem 1.1. **Q.E.D.**

3.1 Proof of Corollary 1.2

The Weyl theorem (see e.g. [12]) states that an n-dimensional Riemannian manifold ($n \geq 4$) is locally conformal flat iff the Weyl curvature tensor W is zero. The vanishing of the Weyl curvature implies that the Cotton-York tensor C^g of the Riemannian metric g is also zero. If $n = 3$, the Weyl tensor is identically zero and the vanishing of the Cotton-York tensor is the necessary and sufficient condition for locally conformal flatness of the metric. In the both cases, (3.9) implies that the conformal Cotton-York tensor C^D of the Einstein-Weyl structure D vanishes. Now, Corollary 1.2 follows from Theorem 1.1 noting that an Einstein manifold with vanishing Weyl tensor is actually of constant sectional curvatures. **Q.E.D.**

Remark 3.1. In the above arguments, the compactness assumption is only used to ensure that the dual vector field T to θ is actually Killing. Theorem 1.1 and Corollary 1.2 hold for $\mathbf{M}$ non-compact with the additional hypothesis that the conformal class $[g]$ contains a metric g such that the dual vector field T to θ is Killing with respect to g.

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