# HERMITIAN STRUCTURES ON TWISTOR SPACES

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**Abstract**

The paper contains description of the orthogonal complex structures with respect to the natural one parameter family of Riemannian metrics on the (negative) twistor space over a self-dual Einstein Riemannian four-manifold. We prove that if the twistor space of a compact self-dual Einstein four-manifold admits more than one orthogonal complex structure then the four-manifold has a Kähler structure. Considering the flag manifold $F_{1,2}$ which is the twistor space of $\mathbf{CP}^2$ endowed with the Fubini-Study metric, we obtain that any invariant Einstein metric on $F_{1,2}$ admits even locally exactly three orthogonal complex structures which are the invariant ones.

## 1 Introduction

Let $(M^{2n}, g)$ be an oriented Riemannian manifold of dimension 2n. A complexstructure on $M$, viewed as an integrable almost-complex structure $J$, is*positive* and *orthogonal* whenever $J$ induces the same orientation on $M$ and $J$ is $g$-skew-symmetric. Then, for any such a structure, the pair $(g, J)$ determines a *positive Hermitian structure* on $M$.

Given a Riemannian manifold $(M^{2n}, g)$ it is natural to ask (see [39])if there exist orthogonal complex structures on $(M, g)$? If so, we would like to describe the set of all orthogonal complexstructures. This question (which in fact concerns the conformal structure determined by $g$) can be asked either locally or globally, and the corresponding questions can be rather different in nature. While on an oriented (real) surface the complex structures and the Riemannian conformal classes coincide, when $n \geq 2$ the *local* existence of orthogonal complex structures imposes constraints on the (conformally invariant) Weyl curvature tensor of $M$ (see for example [40]).

If $M$ is a (real, oriented) four-dimensional manifold then any positive orthogonal complex structure $J$ is *determined*, up to a 4-fold ambiguity, by the self-dual Weyl tensor $W^+$ at any point where it is non-zero. More precisely, $J$ is there equal to some universal function of the eigenforms and the eigenvalues of $W^+$, operating on the bundle of self-dual 2-forms (the above mentioned ambiguity comes from the lack of a canonical orientation for the eigenspaces of $W^+$, cf. [36, 5]). In particular, if $W^+$ is not identically zero, there are (up to sign) at most 2 distinct compatible positive complex structures [34]; here and henceforth, *distinct* means that there is a point where the complex structures are not equal up to sign. A criterion for the (local) existence of an orthogonal complex structure in terms of the covariant derivative of $W^+$ is obtained in [5], where a "Goldberg-Sachs theory" for the Riemannian 4-manifolds is developed.

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A convincing manifestation of the effectiveness of this theory appears in the case when $g$ is Einstein [35, 16, 32, 5]; we have there that the local existence of a positive orthogonal complex structure is equivalent to the degeneracy of the spectrum of $W^+$; if the metric is not anti-self-dual, thenthe compatible complex structure is unique (up to sign) and is globally defined, replacing $M$ by a two fold covering if necessary.

In the case of *compact* anti-self-dual manifold $(M, g)$ (i.e. when $W^+ \equiv 0$) the problems of local and globalexistence of positive orthogonal complex structures are quite different : locally there are infinitely many compatible complex structures [8], while the existence of more than two distinct everywhere, globally defined, compatible positive complex structures forces $(M, g)$ to be hyperhermitian surface [34], i.e. $(M, g)$ is either a K3 surface with a Calabi-Yau Ricci-flat metric, a flat 4-torus, or $S^1 \times S^3$ with a hyperhermitian conformally flat metric [11]. Compact four-manifolds admitting two distinct positive orthogonal complex structures are called *bihermitian surfaces*. In the anti-self-dual case these have been classified by M.Pontecorvo in [34]. For some results concerning the non anti-self-dual case see [6, 28]. As consequence of these works appears the observation that "generically" on a *compact* oriented Riemannian 4-manifold there is at most one *globally defined* compatible positive complex structure.

When the dimension of $M$ is more than 4 the situation is more complicated (see [39] and the included references). For example, consider $\mathbf{CP}^n$, $n \geq 3$ with the Fubini-Study metric. It is well known that the unique globally defined positive orthogonal complex structure is the canonical one while locally there are infinitely many positive orthogonal complex structures since the Bochner tensor of the canonical Hermitian structure of $\mathbf{CP}^n$ vanishes, see [33]. More generally, in the case of a compact Riemannian symmetric space $(M, g)$ the question of global existence of orthogonal complex structures has been successfully studied by Burstall and Rawnsley[12] by introducing the twistor space of $M$. If $(M, g)$ isa compact inner-symmetric Riemannian manifold then $M$ admits a global orthogonal complex structure if and only if $M$ is a Hermitian symmetric space ; in this case the only orthogonal complex structure is the invariant one, cf. [13]. Using different approach Gauduchon has obtained the same result if $M$ is a compact quotient of an irreducible Riemannian symmetric space of non-compact type [19]. The existence of complex structures on the *quaternionic* manifolds (of dimension $4n, n \geq 2$) has been recently studied in [3, 4]. Unfortunately, for a general Riemannian manifold a little is known for the set of orthogonal complex structures.

The main concern of the present paper is the local existence of orthogonal complex structures on some special type 6-dimensional Riemannian manifolds, namely the twistor spaces of oriented self-dual Einstein 4-manifolds. The (*negative*) twistor space of an oriented Riemannian 4-manifold $M$ is the 2-sphere bundle $\mathcal{Z}$ over $M$ consisting of the unit anti-self-dual two forms on $M$. The 6-manifold $\mathcal{Z}$ admits a natural 1-parameter family of Riemannian metrics $h_t$, $t > 0$, which are firstly defined and studied by Friedrich and Kurke [21]. When $(M, g)$ is an oriented self-dual Riemannian 4-manifold the canonical almost-complex structure $\mathbf{J}$ on $\mathcal{Z}$ (which is orthogonal with respect to any metric $h_t$) is integrable [8] and defines a 1-parameter family of Hermitian structures $(h_t, \mathbf{J})$ on $\mathcal{Z}$.

If moreover $(M, g)$ is Einstein, the O'Neill formulas give an expression of Riemannian curvature of $(\mathcal{Z}, h_t)$ in terms of the curvature of $(M, g)$ (see [20, 27, 43, 14, 15]). In section 3 below we make use of the special form of the curvature of $(\mathcal{Z}, h_t)$ given in [14] which allows us to relate the orthogonal complex structures on $(\mathcal{Z}, h_t)$ to these on $(M, g)$.

Particular case of our considerations is the flag manifold $F_{1,2} = \mathbf{U}(3)/\mathbf{U}(1) \times \mathbf{U}(1) \times \mathbf{U}(1)$, which is the twistor space of $\mathbf{CP}^2$ endowed with the Fubini-Study metric (of constant holomorphic sectional curvature 4) and where $h_t$ is a one-parameter family of $\mathbf{U}(3)$ - *invariant* metrics (see Example 2). In Theorem 1 we obtain that for any $t > 0$ the metric $h_t$ has exactly three commuting orthogonal complex structures (which are the *invariant* complex structures on $F_{1,2}$). According to [20] the metrics $h_{\frac{1}{2}}$ and $h_{\frac{1}{4}}$ are Einstein and it is well known that any invariant Einstein metric on $F_{1,2}$ is equivalent to one of them (see for example [7]). In particular, this gives that for any such a metric there are exactly three (invariant) commuting orthogonal complex structures (Corollary2 bellow). As a consequence of the classification of Fano 3-folds of index at least 2 the same is true for any Kähler-Einstein metric of positive type with respect to arbitrary complex structure on $F_{1,2}$ see Corollary 3.

As another application of the results in section 3 we provide on $\mathbf{CP}^3$ (considered as the twistor space of $S^4$) a Bochner-flat non-Kähler almost Hermitian structure in the class $\mathcal{AH}_2$, see Example 1.

Finally, using the *Riemannian Goldberg-Sachs Theorem* [5, 32, 35], we show in Theorem 2 that the twistor space $(\mathcal{Z}, h_t)$ of a *compact* self-dual Einstein 4-manifold $(M, g)$ has an orthogonalcomplex structure different from $\pm\mathbf{J}$ for *any* $t > 0$ iff $(M, g)$ is isometrically equivalent to one of the following: $\mathbf{CP}^2$ with the Fubini-Study metric; a compact flat 4-manifold; a K3-surface or its $\mathbf{Z}_2$ and $\mathbf{Z}_2 \times \mathbf{Z}_2$-quotients endowed with the opposite orientation and a Calabi-Yau metric; a compact quotient of the complex hyperbolic space with the Bergman metric.

## 2 Positive orthogonal complex structures and twistor spaces - Preliminary results

Let $(M, g)$ be a $2n$- dimensional oriented Riemannian manifold. We wish to study positive orthogonal (almost) complex structures on $(M, g)$, which we view as sections of the fibre bundle $\pi_+ : Z^+M = P \times_{O(2n)} (SO(2n)/U(n)) \longmapsto M$, where $P \mapsto M$ denote the canonical principal $O(2n)$-bundle. The vertical distribution $\mathcal{V} =\mathrm{Ker}(\pi_+)_*$ inherits a canonical complex structure$J^{\mathcal{V}}$ since the fibre $H_n = SO(2n)/U(n)$ is a Hermitian-symmetric space.Moreover, the Levi-Civita connection $\nabla$ on $M$ induces a splitting$TZ^+M = \mathcal{H} \oplus \mathcal{V}$of the tangent bundle of $Z^+M$ into horizontal and vertical components, sothat $\mathcal{H} \cong (\pi_+)^{-1}TM$ acquires a tautological complex structure$J^{\mathcal{H}}$ given by $J^{\mathcal{H}}_{x,j} = j$ for $x \in M$ and$j \in (\pi_+)^{-1}(x) \cong H_n$.

Following [8, 17], we define two almost-complex structures $\mathbf{J}$ and $\mathbf{I}$ on $Z^+M$ by

$$\mathbf{J} = J^{\mathcal{H}} + J^{\mathcal{V}}; \ \mathbf{I} = J^{\mathcal{H}} - J^{\mathcal{V}}.$$

It is well known that

(a) $(2n = 4)$ $\mathbf{J}$ *is integrable iff the positive Weyl tensor* $W^+$ *of* $(M, g)$ *vanishes, i.e.* $g$ *is an anti-self-dual metric* [8],

(b) $(2n > 4)$ $\mathbf{J}$ *is integrable iff the Weyl tensor* $W$ *of* $(M, g)$ *vanishes, i.e.* $g$ *is a locally conformally flat metric* [33],

(c) $\mathbf{I}$ *is never integrable [17].*

In [36], Salamon shows that integrability of a positive orthogonal almost complex structure $J$ of $(M, g)$ is equivalent to the holomorphicity of $J$ viewed as a map $J : (M, J) \mapsto (Z^+M, \mathbf{J})$. In fact one can say more. In [33], the Nijenhuis tensor $N^{\mathbf{J}}$ is calculated and we have:

**Proposition 1** *Let* $\sigma = (x, j) \in Z^+M$ *with* $(1,0)$*-space* $T^{(1,0)}_x(j) \subset T_xM \otimes \mathbf{C}$. *Denote by* $R$ *the Riemannian curvature tensor of* $M$. *Then the Nijenhuis tensor* $N^{\mathbf{J}}$ *vanishes at* $\sigma$ *iff*

$$R_x(T^{1,0}(j), T^{1,0}(j))T^{1,0}(j) \subset T^{1,0}(j). \tag{1}$$

Denote by $Z_0^+M$ the zero-set of $N^{\mathbf{J}}$. Since any positive orthogonal complex structure $J$ satisfies (1) we have that $J$ lies entirely in $Z_0^+M$.

In the case when $(M,g)$ is a 4-dimensional oriented Riemannian manifold the twistor fiber $H_2$ is isomorphic to the two-sphere and the set $\pi^{-1}(x)\cap Z_0^+M$ either consists of 4 or 2 (antipodal) points or it is whole $S^2$-fibre, according to the spectrum of the positive Weyl tensor operating on the bundle of self-dual 2-forms. More precisely we have [38, 39, 5]

**Proposition 2** *Let $(M,g)$ be an oriented Riemannian 4-manifold and let $\lambda_- \leq \lambda_0 \leq \lambda_+$ be the eigenvalues of of the positive Weyl tensor $W^+$, considered as an endomorphism of the vector bundle $\Lambda^+(M)$ of self-dual two-forms on $M$. Then we have:*

(i) *if $W^+$ is non-degenerate at $x \in M$ then $\pi^{-1}(x) \cap Z_0^+M$ consists of 2 (determined up to sign ) different positive orthogonal almost complex structures $J'$ and $J''$ whose Kähler forms $\Omega'$ and $\Omega''$ are respectively given by:*

$$\Omega' = \frac{(\lambda_+ - \lambda_0)^{\frac{1}{2}}}{(\lambda_+ - \lambda_-)^{\frac{1}{2}}}\Omega_- - \frac{(\lambda_0 - \lambda_-)^{\frac{1}{2}}}{(\lambda_+ - \lambda_-)^{\frac{1}{2}}}\Omega_+,$$

*and*

$$\Omega'' = \frac{(\lambda_+ - \lambda_0)^{\frac{1}{2}}}{(\lambda_+ - \lambda_-)^{\frac{1}{2}}}\Omega_- + \frac{(\lambda_0 - \lambda_-)^{\frac{1}{2}}}{(\lambda_+ - \lambda_-)^{\frac{1}{2}}}\Omega_+,$$

*where $\Omega_+$ (resp. $\Omega_-$) stands for a generator of the eigenspace of $W^+$ with respect to $\lambda_+$, (resp. $\lambda_-$), normalized by $|\Omega_+|^2 = |\Omega_-|^2 = 2$.*

(ii) *if $W^+$ is degenerate but non zero at $x \in M$ then there is one (up to sign) element $J \in (\pi_+)^{-1}(x) \cap Z_0^+M$ whose Kähler form $\Omega$ is determined at $x$ as being the generator of the simple eigenspace of $W^+$ with square-norm equal to 2;*

(iii) *if at point $x \in M$ the positive Weyl tensor vanishes, then any element of the fibre $(\pi_+)^{-1}(x)$ belongs to $Z_0^+$.*

Note that a section $J$ of $Z_0^+M$ may or may not be integrable. In the case when $(M,g)$ is Einstein 4-manifold, or more generally, when the positive Weyl tensor is harmonic, a Riemannian version of Goldberg-Sachs theorem [35, 32, 5] asserts that there is a positive orthogonal complex structure $J$ if and only if the spectrum of $W^+$ is degenerate. Moreover, according to a result of Derdzinski [16], an oriented Einstein 4-manifold with degenerate positive Weyl tensor at any point is either anti-self-dual or the positive Weyl tensor never vanishes. Summarizing, we have the following:

**Proposition 3** *Let $(M,g)$ be an oriented Einstein Riemannian 4-manifold. Then the following three conditions are equivalent:*

(i) *the spectrum of $W^+$ is degenerate;*

(ii) *there exists an orthogonal, positive, integrable almost complex structure $J$ in the neighborhood of some ( or equivalently, of each ) point of $M$;*

(iii) *$(M,g)$ is either anti-self-dual, or $W^+$ has two distinct eigenvalues at any point of $M$.*

We now focus our attention to the 6-dimensional manifolds. Since the positive and the negative twistor spaces of oriented 6-dimensional Riemannian manifold $(M,g)$ can be identified via the map $j \mapsto -j$ on the fibre, we will consider the twistor space $Z(M,g)$ as the quotient space of the principle $O(6)/U(3)$-bundle of all orthogonal almost complex structures of $(M,g)$ under this action. It is well known that the fiber $H_3 = SO(6)/U(3)$ is isomorphic to $\mathbf{CP}^3$ and we will make use of the explicit identification given in [1] (for more details see also [37, 44, 25]). Let $V$

be a complex 4-dimensional vector space endowed with a Hermitian inner product $h$ and with a volume form $\Phi \in \Lambda^4 V$. The Hodge $*$-operator $*$ is defined on $\Lambda^2 V$ by

$$\xi \wedge *\eta = h(\xi, \eta)\Phi.$$

Thus the $\mathbf{C}$-anti-linear endomorphism $*$ induces on the 6-dimensional complex vector space $\Lambda^2 V$ a real structure. The fixed points set of $*$ forms a real 6-dimensional vector space $W$ with a positive definite inner product $g$ coming from $h$. For any $[v] \in \mathbf{P}(V)$ we have the $h$-orthogonal splitting:

$$\Lambda^2 V = V_v \oplus V_v^{\perp}, \tag{2}$$

where $V_v$ denotes the vector space generated of all 2-vectors $v \wedge u$ with $h(u, v) = 0$ and $V_v^{\perp}$ is the orthogonal component of $V_v$ in $\Lambda^2 V$. Notice that $V_v^{\perp}$ is spanned by the 2-vectors $u' \wedge u''$ with $h(u', v) = h(u'', v) = 0$. The splitting (2) of $\Lambda^2 V = W \otimes \mathbf{C}$ defines a positive $g$-orthogonal complex structure on the vector space $W$ with (1,0) and (0,1)-spaces equal to $V_v$ and $V_v^{\perp}$, respectively. This gives the identification of $\mathbf{P}(V) \cong \mathbf{CP}^3$ with the space $SO(6)/U(3)$ [1, Lemma 4.1]. In terms of this correspondence we have the following

**Lemma 1** *Two different positive orthogonal complex structures $J'$ and $J''$ on $(W, g)$ commute if and only if the corresponding $[v']$ and $[v''] \in \mathbf{P}(V)$ are $h$-orthogonal.*

*Proof.* If we assume that $h(v', v'') = 0$ then the non-zero 2-vector $Z = v' \wedge v''$ belongs to the (1,0)-space of both $J'$ and $J''$. The (complex) 2-dimensional subspace $V_Z = span\{v' \wedge u : h(v', u) = h(v'', u) = 0\}$ of $V_{v'}$ belongs to the (1,0)-space of $J'$ and to the (0,1)-space of $J''$, hence $J'$ and $J''$ commute. Conversely, any two commuting positive orthogonal complex structures $J'$ and $J''$ on $(W, g)$ have a common (1,0)-vector, say $Z$. Then $h$-orthogonal complement $Z^{\perp}$ of $Z$ in $T^{1,0}_{J'}$ (withrespect to the appropriate extension of $h$ to $\Lambda^2 V$) lies entirely in $T^{0,1}_{J''}$. Letting $Z = v' \wedge u'$ for any $v$ with $h(v', v) = h(u', v) = 0$, we have $h(v' \wedge v, v' \wedge u') = 0$ and since $v' \wedge v \in T^{1,0}_{J'} \cap T^{0,1}_{J''}$ we infer $v' \in V_{v''}^{\perp}$. $\square$

As immediate consequence we obtain

**Corollary 1** *Let $\{J_0, J_1, J_2, J_3\}$ be orthogonal complex structures on $(W, g)$. Then $J_i, i = 0, ..., 3$ mutually commute iff the corresponding generators $v_i, i = 0, ..., 3$ in $\mathbf{P}(V)$ can be chosen in such a way that $\{v_0, v_1, v_2, v_3\}$ is a unitary frame of $(V, h)$ and $v_0 \wedge v_1 \wedge v_2 \wedge v_3 = \Phi$.*

Fix an orthogonal complex structure $J_0$ of $(W, g)$ which corresponds to $[v_0] \in \mathbf{P}(V)$ and let $\{v_0, v_1, v_2, v_3\}$ be a unitary frame of $(V, h)$ with $v_0 \wedge v_1 \wedge v_2 \wedge v_3 = \Phi$. Then the vectors

$$Z_1 = v_0 \wedge v_1, \ Z_2 = v_0 \wedge v_2, \ Z_3 = v_0 \wedge v_3,$$

give a unitary frame of $T^{1,0}_{J_0}$ ; their complex-conjugated vectors are

$$\overline{Z}_1 = v_2 \wedge v_3, \ \overline{Z}_2 = -v_1 \wedge v_3, \ \overline{Z}_3 = v_1 \wedge v_2.$$

Denote by $\{J_0, J_1, J_2, J_3\}$ the corresponding commuting positive orthogonal complex structures on $(W, g)$. It is easily seen that the $(1,0)$-spaces of $J_0, J_1, J_2, J_3$ are respectively spanned by the triples:

$$\{Z_1, Z_2, Z_3\}, \{Z_1, \overline{Z}_2, \overline{Z}_3\}, \{\overline{Z}_1, Z_2, \overline{Z}_3\}, \{\overline{Z}_1, \overline{Z}_2, Z_3\}.$$

For an element $[v] \in \mathbf{P}(V)$ with homogeneous coordinates $[\alpha_0, \alpha_1, \alpha_2, \alpha_3]$, the corresponding positive orthogonal complex structure $J$ is determined by its (1,0)-space spanned by the vectors:

$$\begin{array}{rcl} Z_0^J &=& -\alpha_1 Z_1 - \alpha_2 Z_2 - \alpha_3 Z_3; \\ Z_1^J &=& \alpha_0 Z_1 - \alpha_2 \overline{Z}_3 + \alpha_3 \overline{Z}_2; \\ Z_2^J &=& \alpha_0 Z_2 + \alpha_1 \overline{Z}_3 - \alpha_3 \overline{Z}_1; \\ Z_3^J &=& \alpha_0 Z_3 - \alpha_1 \overline{Z}_2 + \alpha_2 \overline{Z}_1. \end{array} \tag{3}$$

The only relation among them is $\Sigma_{j=0}^3 \alpha_j Z_j^J = 0$.

We can decompose $\mathbf{CP}^3$ as

$$\mathbf{CP}^3 \cong \mathbf{C}^3 \cup \mathbf{C}^2 \cup \mathbf{CP}^1, \tag{4}$$

where:
(a) the copy of $\mathbf{CP}^1$ consists of the elements of $\mathbf{CP}^3$ with homogeneous coordinates $[0, \delta_1, \delta_2, 0]$; the corresponding $\mathbf{CP}^1$-family of positive orthogonal complex structures on $(W, g)$ consists of all elements which commute with but differ from both $J_0$ and $J_3$.
(b) the copy of $\mathbf{C}^2$ in $\mathbf{CP}^3$ is determined by the elements with homogeneous coordinates $[0, \gamma_1, \gamma_2, 1]$, i.e. the corresponding family of positive orthogonal complex structures on $(W, g)$ consists of all elements which commute with $J_0$ but do not commute with $J_3$. The $(1, 0)$-space of these complex structures is spanned by the complex vectors

$$\overline{Z}_2 - \gamma_2 \overline{Z}_3; \ \ \overline{Z}_1 - \gamma_1 \overline{Z}_3; \ \ Z_3 + \gamma_1 Z_1 + \gamma_2 Z_2. \tag{5}$$

(c) the copy of $\mathbf{C}^3$ is determined by the elements of $\mathbf{CP}^3$ with homogeneous coordinates $[1, \beta_1, \beta_2, \beta_3]$, i.e. which have a $(1, 0)$-space spanned by the complex vectors

$$Z_1 + \beta_3 \overline{Z}_2 - \beta_2 \overline{Z}_3; \ \ Z_2 - \beta_3 \overline{Z}_1 + \beta_1 \overline{Z}_3; \ \ Z_3 + \beta_2 \overline{Z}_1 - \beta_1 \overline{Z}_2. \tag{6}$$

Equivalently, the family (c) consists of all positive orthogonal almost-complex structures which do not commute with $J_0$.

**Remark 1** It follows from Proposition 1 and (3) that at any $x \in M$ $Z_0(M, g) \cap \pi^{-1}(x)$ is the zero set of at most six independent homogeneous polynomials on $\mathbf{CP}^3$ of degree 4. In particular it is complex algebraic subset of $\mathbf{CP}^3$

# 3 Hermitian structures on the twistor space of a self-dual Einstein four-manifold

Suppose from now on that $(M, g)$ is an oriented, self-dual, Einstein, Riemannian 4-manifold. Denote by $\mathcal{Z}$ the (negative) twistor space of $M$. Besides the almost complex structures $\mathbf{I}, \mathbf{J}$ the splitting of $T\mathcal{Z}$ into horizontal and vertical components determines 1-parameter family of Riemannian metrics $h_t$ on $\mathcal{Z}$ (see [21]) defined by

$$h_t = (\pi_-)^* g + t g^{\mathcal{V}}, \ t > 0,$$

where $\pi_-$ is the twistor projection, $g$ is the metric of $M$ and $g^{\mathcal{V}}$ is the standard metric (of constant curvature 1) on the fibre $H_2 \cong \mathbf{CP^1}$.

By the O'Neill formulas for the Riemannian submersion $\pi_- \ : (\mathcal{Z}, h_t) \to (M, g)$ whose fibers are totally geodesic submanifolds of $(\mathcal{Z}, h_t)$ one can calculate the Riemannian curvature tensor

$R^t$ of $(\mathcal{Z}, h_t)$, cf. [21, 20, 27, 43, 14, 15] . We shall use the following formula [14, 15] : if $E, F, G, H \in T_\sigma \mathcal{Z}$ and $X = \mathcal{H}E,\ Y = \mathcal{H}F,\ Z = \mathcal{H}G,\ T = \mathcal{H}H,\ A = \mathcal{V}E,\ B = \mathcal{V}F,\ C = \mathcal{V}G,\ D = \mathcal{V}H$ denotes respectively the horizontal and vertical components of $E, F, G, H$ then

$$
\begin{aligned}
& R^t_\sigma(E,F,G,H) = R^M_{\pi_-(\sigma)}((\pi_-)_*X, (\pi_-)_*Y, (\pi_-)_*Z, (\pi_-)_*T) \qquad (7)\\
- \ & 2[t(\frac{s}{24})^2 - \frac{s}{24}][h_t(\mathbf{J}X,Y)h_t(\mathbf{J}C,D) + h_t(\mathbf{J}Z,T)h_t(\mathbf{J}A,B)]\\
+ \ & [t(\frac{s}{24})^2 - \frac{s}{24}][h_t(\mathbf{J}X,T)h_t(\mathbf{J}B,C) + h_t(\mathbf{J}Y,Z)h_t(\mathbf{J}A,D)\\
- \ & h_t(\mathbf{J}X,Z)h_t(\mathbf{J}B,D) - h_t(\mathbf{J}Y,T)h_t(\mathbf{J}A,C)]\\
+ \ & t(\frac{s}{24})^2[h_t(X,Z)h_t(B,D) - h_t(X,T)h_t(B,C)\\
+ \ & h_t(Y,T)h_t(A,C) - h_t(Y,Z)h_t(A,D)]\\
+ \ & t(\frac{s}{24})^2[h_t(\mathbf{J}X,Z)h_t(\mathbf{J}Y,T) - h_t(\mathbf{J}X,T)h_t(\mathbf{J}Y,Z)\\
+ \ & 2h_t(\mathbf{J}X,Y)h_t(\mathbf{J}Z,T)]\\
- \ & 3t(\frac{s}{24})^2[h_t(X,Z)h_t(Y,T) - h_t(Y,Z)h_t(X,T)]\\
+ \ & \frac{1}{t}[h_t(A,C)h_t(B,D) - h_t(A,D)h_t(B,C)].
\end{aligned}
$$

The metrics $h_t$ are compatible with both almost-complex structures $\mathbf{I}$, $\mathbf{J}$ and the almost Hermitian 6-manifolds $(\mathcal{Z}, h_t, \mathbf{I})$ and $(\mathcal{Z}, h_t, \mathbf{J})$ belong to the class $\mathcal{AH}_2$, see [15]. In particular, both $\mathbf{I}$ and $\mathbf{J}$ satisfy (1), i.e. $\mathbf{I}$ and $\mathbf{J}$ lie in the zero set $Z_0(\mathcal{Z}, h_t)$ of the twistor space $Z(\mathcal{Z}, h_t)$ of $(\mathcal{Z}, h_t)$. Set $\mathcal{J}_0 = \mathbf{J}$ and $\mathcal{J}_3 = -\mathbf{I}$. Then we can decompose the fibre $H_3 \cong \mathbf{CP}^3$ of the twistor bundle $\pi : Z(\mathcal{Z}, h_t) \mapsto (\mathcal{Z}, h_t)$ as in the preceding section. For any point $\sigma = (x, j)$ of $\mathcal{Z}$ the elements of $\mathbf{CP}^1$-family (a) of positive orthogonal complex structures on $(T_\sigma\mathcal{Z}, h_t)$ coincide on the vertical component of $T_\sigma\mathcal{Z}$, so these can be identified with the fibre of positive twistor space $\pi_+ : Z^+(M) \mapsto M$ over $x \in M$ using the map:

$$
\begin{array}{rcll}
J \in \pi_+^{-1}(x) & \mapsto & \mathcal{J} \in \pi^{-1}(\sigma) & \\
\mathcal{J}X^h & = & JX & \forall X \in TM\\
\mathcal{J}A & = & -J^{\mathcal{V}}A & \forall A \in \mathcal{V}
\end{array} \qquad (8)
$$

where $X^h$ denotes the horizontal lift of $X$.
Moreover, we have the following

**Lemma 2** *Let $J$ be a positive orthogonal almost complex structure on $(M, g)$ and let $\mathcal{J}$ be the corresponding lift to an orthogonal almost complex structure on $(\mathcal{Z}, h_t)$. Then*
(i) *$\mathcal{J}$ belongs to $Z_0(\mathcal{Z}, h_t)$ iff $J$ itself satisfies (1).*
(ii) *$\mathcal{J}$ is integrable iff $(M, g, J)$ is either a flat Hermitian surface or a Kähler-Einstein surface.*

*Proof :* The first part is immediate consequence of (7) and the fact that $\mathcal{J}$ and $\mathbf{J}$ commute. To prove the second statement we shall use the O'Neill formulas for the Riemannian submersion $\pi_- : (\mathcal{Z}, h_t) \to (M, g)$ and the fact that the fibers are totally geodesic submanifolds of $(\mathcal{Z}, h_t)$. Thus if we denote by $D^t$ the Levi-Civita connection of $(\mathcal{Z}, h_t)$ then we have at any point $\sigma = (x, j)$ of $\mathcal{Z}$ (see [14, Lemma 3.1]) :

$$(D^t_{X^h} Y^h)_\sigma = (\nabla_X Y)^h_\sigma + \frac{1}{2} R_x(X, Y) j, \qquad (9)$$

$$(D^t_A Y^h)_\sigma = -\frac{t}{2}(R_x(J^{\mathcal{V}}A)X)^h_\sigma, \tag{10}$$

where $X, Y$ are vector fields on $M$ ; $X^h, Y^h$ are their horizontal lifts ; $A$ is an arbitrary vertical vector field on $\mathcal{Z}$ and $R_x(X,Y)j$ denotes the $j$-anti-invariant two form $R_x(X,Y)(j.,.) + R_x(X,Y)(.,j.)$ at $x$ viewed as an element of $\mathcal{V}_\sigma$. Now it is easily seen that $D^t$ preserves $T^{1,0}_{\mathcal{J}}\mathcal{Z}$ if and only if $\nabla$ preserves $T^{1,0}_J M$ *and* for any $(1,0)$ vectors $X, Y$ we have $R(X \wedge Y) \in \Lambda^- M$. The first condition is equivalent to $J$ being integrable. Let $R = \frac{s}{12}Id + B + W^+ + W^-$ be the $SO(4)$-splitting of the curvature operator $R$. It is well known that the traceless Ricci operator $B$ switches $\Lambda^+ M$ and $\Lambda^- M$, so since for any $(1,0)$-vectors $X, Y$ the two-vector $X \wedge Y$ belongs to $\Lambda^+ M \otimes \mathbf{C}$, the second condition reads $\frac{s}{12} X \wedge Y + W^+(X \wedge Y) = 0$ for any $(1,0)$-vectors $X, Y$. It follows from [5, Lemma 1] (see also [42]) that the latter is equivalent to $W^+$ being degenerate *and* $\kappa = s$, where $k = 3(W^+(\Omega), \Omega)$ is the conformal scalar curvature of $(g, J)$ ; $\Omega$ is the Kähler form of $(g, J)$. In particular, in the Einstein case, we have $|W^+| = const$ (see [5, Lemma 1]) and according to [5, Proposition 1] either $s \equiv 0$ or $(g, J)$ is Kähler. $\square$

**Remark 2** The proof of Lemma 2 shows that for any Kähler surface $(M, g, J)$ (which is not necessary Einstein) the almost complex structure $\mathcal{J}$ on $\mathcal{Z}$ is integrable and the converse is true provided that $M$ is compact (see [42]). Moreover, if we denote by $\Sigma_-$, $K^{\frac{1}{2}}$ and $T$ the (locally defined) bundle of negative spinors, a (locally defined) square-root of the canonical line bundle $K$ and the holomorphic tangent bundle of $(M, J, g)$, respectively, then $\mathcal{J}$ is the canonical complex structure coming from the identifications

$$\mathcal{Z} \cong \mathbf{P}(\Sigma) \cong \mathbf{P}(T \otimes K^{\frac{1}{2}}),$$

see [41]. *(End of Remark 2).*

If $J$ is a *negative* orthogonal almost complex structure on $(M, g)$ we can use (8) to lift $J$ to an almost complex structure $\mathcal{J}$ on $\mathcal{Z}$, compatible with $h_t$ for any $t > 0$. On the analogy of Lemma 2 we obtain

**Lemma 3** *Let $J$ be a* negative *orthogonal almost complex structure on $(M, g)$ and let $\mathcal{J}$ be the corresponding lift to an orthogonal almost complex structure on $(\mathcal{Z}, h_t)$. Then we have the following equivalences:*

$$\begin{aligned} &\mathcal{J} \textit{ belongs to } Z_0(\mathcal{Z}, h_t) &&\Leftrightarrow \quad s \equiv 0; \\ &\mathcal{J} \textit{ is integrable} &&\Leftrightarrow \quad s \equiv 0 \textit{ and } J \textit{ is integrable.} \end{aligned}$$

*Proof :* If $s \equiv 0$, then from (7) we have that $\mathcal{J}$ satisfies (1) iff $J$ itself satisfies (1). The latter condition is true for any negative orthogonal almost complex structure since the negative Weyl tensor vanishes. Conversely, suppose that $\mathcal{J}$ satisfies (1) at any point of $\mathcal{Z}$. Choose $\sigma = (x, j)$ with $j$ anti-commuting with $J_x$. We then obtain from (7) that (1) is equivalent to $s \equiv 0$. We conclude the second part as in the proof of Lemma 2-(ii). $\square$

Let $J$ be a positive orthogonal almost complex structure on $(M, g)$ satisfying condition (1) (i.e. $J$ is determined according to Proposition 2) and let $\mathcal{J}_1$ and $\mathcal{J}_2$ be the almost Hermitian structures on $(\mathcal{Z}, h_t)$ which are the lifts of $J$ and $-J$ by (8), respectively. Thus $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2$ and $\mathcal{J}_3$ all satisfy (1) and mutually commute. Using the quadruple $\{\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$ we can introduce the homogeneous coordinates on the fibre $\mathbf{CP}^3$ as in the above section (with respect to these coordinates $\mathcal{J}_0 = [1,0,0,0]; \mathcal{J}_3 = [0,0,0,1]; \mathcal{J}_1 = [0,1,0,0]; \mathcal{J}_2 = [0,0,1,0]$). Denote by $\{Z_1, Z_2, Z_3, \overline{Z}_1, \overline{Z}_2, \overline{Z}_3\}$ the corresponding frame of the $T_\sigma\mathcal{Z}\otimes\mathbf{C}$. Notice that the complex vectors $Z_1$, $Z_2$ are horizontal while $Z_3$ belongs to the vertical component of $T_\sigma\mathcal{Z}\otimes\mathbf{C}$. By (7), (5) and (6) we easily determine the positive orthogonal almost-complex structures in the families (b) and (c) which satisfy (1) : let $s$, $\Omega(.,.)=g(J.,.)$ and $\kappa = 3(W^+(\Omega),\Omega)$ be the scalar curvature, Kähler form and the conformal scalar curvature of $(M, J, g)$ and set

$$P = [-t(\frac{s}{12})^2 + \frac{s}{6} - \frac{\kappa}{12} - \frac{1}{t}];$$

$$Q = (\pi_-)^*(W^+(\Omega))(Z_1, \overline{Z}_2);$$

$$L = [2t(\frac{s}{12})^2 - \frac{s}{3} - \frac{\kappa}{12} - \frac{1}{t}],$$

then we have the following:

**Lemma 4** *An orthogonal almost complex structure in the family* (*b*) *belongs to* $Z_0(\mathcal{Z}, h_t)$ *if and only if its homogeneous coordinates* $\gamma_1$, $\gamma_2$ *satisfy*

$$\begin{aligned} P(\gamma_1)^2 + Q\gamma_1\gamma_2 &= 0; \\ P(\gamma_2)^2 - \overline{Q}\gamma_1\gamma_2 &= 0; \\ \overline{Q}(\gamma_1)^2 - Q(\gamma_2)^2 + 2(P + \frac{\kappa}{6})\gamma_1\gamma_2 &= 0. \end{aligned}$$

**Lemma 5** *An orthogonal almost complex structure in the family* (*c*) *belongs to* $Z_0(\mathcal{Z}, h_t)$ *if and only if its homogeneous coordinates* $\beta_1$, $\beta_2$, $\beta_3$ *satisfy*

$$\begin{aligned} s\beta_3 &= 0; \\ L(\beta_1)^2 + \overline{Q}\beta_1\beta_2 &= 0; \\ L(\beta_2)^2 - Q\beta_1\beta_2 &= 0; \\ -Q(\beta_1)^2 + \overline{Q}(\beta_2)^2 - (L + \frac{\kappa}{4})\beta_1\beta_2 &= 0. \end{aligned}$$

We are now ready to describe the set $Z_0(\mathcal{Z}, h_t)$. First of all notice that from Propositions 2 and 3 and using the analyticity of $g$, the following three possibilities appears for a self-dual Einstein Riemannian 4-manifold $(M, g)$:

(A) The positive Weyl tensor $W^+$ is identically zero, i.e. $(M, g)$ is an oriented 4-manifold of constant sectional curvature $\frac{s}{12}$.

(B) The positive Weyl tensor $W^+$ has exactly two different eigenvalues at any point, i.e. by replacing $M$ by a two-fold covering if necessary $M$ is a self-dual Einstein Hermitian surface. If moreover $M$ is compact, then $M$ is a complex space form by a result of Boyer [11].

(C) The positive Weyl tensor is non degenerate in a dense open subset of $M$.

In the notations above we have : $Q \equiv 0$ iff the spectrum of $W^+$ is degenerate; $k \equiv 0$ & $Q \equiv 0$ iff $W^+ \equiv 0$, cf. [38, 5]. Substituting $Q = 0, \kappa = 0$ into lemmas 4 and 5 and using lemmas 2 and 3, we obtain for case (A)

**Proposition 4** *Let $(M,g)$ be an oriented 4-manifold of constant sectional curvature $\frac{s}{12}$. Then:*

(i) *if $\frac{st}{12} = 1 + \sqrt{\frac{3}{2}}$ or $\frac{st}{12} = 1 - \sqrt{\frac{3}{2}}$ then $Z_0(\mathcal{Z}, h_t)$ has two connected components which are $\pm\mathcal{J}_3$ and the $\mathbf{CP}^2$-bundle over $\mathcal{Z}$ of all positive orthogonal almost complex structures of $(\mathcal{Z}, h_t)$ which commute with, but differ from $\pm\mathcal{J}_3$;*

(ii) *if $st = 12$ then $Z_0(\mathcal{Z}, h_t)$ has two connected components which are $\pm\mathcal{J}_0$ and the $\mathbf{CP}^2$-bundle over $\mathcal{Z}$ of all positive orthogonal almost complex structures of$(\mathcal{Z}, h_t)$ which commute with, but differ from $\pm\mathcal{J}_0$;*

(iii) *if $s = 0$, i.e. $(M,g)$ is flat, then for any $t > 0$ the zero-set $Z_0(\mathcal{Z}, h_t)$ is a $\mathbf{CP}^1 \times \mathbf{CP}^1$-bundle over $\mathcal{Z}$ of the lifts of the orthogonal (positive and negative ) almost complex structures on $(M,g)$;*

(iv) *in any other cases $Z_0(\mathcal{Z}, h_t)$ has three connected components: $\pm\mathcal{J}_0$; $\pm\mathcal{J}_3$ and the $\mathbf{CP}^1$-bundle over $\mathcal{Z}$ of the lifts of the positive orthogonal almost complex structures of $(M,g)$.*

**Example 1** According to the results in [33], in the case (i) of the above proposition the almost Hermitian manifold $(\mathcal{Z}, h_t, \mathcal{J}_3 = -\mathbf{I})$ has vanishing Bochner tensor (for the definition of the Bochner tensor of an almost Hermitian manifold see [40]). As already mentioned, the almost Hermitian structure $(h_t, \mathbf{I})$ belongs to the class $\mathcal{AH}_2$ [14]. Thus, $(\mathcal{Z}, h_t, \mathbf{I})$ with $\frac{st}{12} = 1 + \sqrt{\frac{3}{2}}$ is a *non-Kähler* Bochner-flat almost Hermitian structure of class $\mathcal{AH}_2$ on $\mathbf{CP}^3$.

We also observe that in case (ii) of Proposition 4 $(\mathcal{Z}, h_t, \mathcal{J}_0)$ is a Kähler Bochner-flat manifold (which is also of constant holomorphic sectional curvature, see [14]). *(End of Example 1.)*

When $(M, J, g)$ is a self-dual Kähler-Einstein surface we have $Q \equiv 0$ and $\kappa = s$, cf. [18, 16]. In this case lemmas 4, 5, 2 and 3 give

**Proposition 5** *Let $(M, J, g)$ be a self-dual Kähler-Einstein surface. Then:*

(i) *if $\frac{st}{12} = \frac{5+\sqrt{33}}{4}$ or $\frac{st}{12} = \frac{5-\sqrt{33}}{4}$, then $Z_0(\mathcal{Z}, h_t)$ consists of $\pm\mathcal{J}_3$ and two $\mathbf{CP}^1$-bundles $\mathcal{P}_3$ and $\mathcal{P}_4$ determined as the orthogonal almost complex structures of $(\mathcal{Z}, h_t)$ which commute with both $\mathcal{J}_3$ and $\mathcal{J}_1$ (resp. $\mathcal{J}_3$ and $\mathcal{J}_2$ ) but differ from $\pm\mathcal{J}_3$ and $\pm\mathcal{J}_1$ (resp. from $\pm\mathcal{J}_3$ and $\pm\mathcal{J}_2$);*

(ii) *if $\frac{st}{12} \neq \frac{5+\sqrt{33}}{4}$ and $\frac{st}{12} \neq \frac{5-\sqrt{33}}{4}$ then $Z_0(\mathcal{Z}, h_t)$ consists of $\pm\mathcal{J}_0$, $\pm\mathcal{J}_1$, $\pm\mathcal{J}_1$ and $\pm\mathcal{J}_3$.*

**Example 2** Let $F_{1,2} = \mathbf{U}(3)/\mathbf{U}(1) \times \mathbf{U}(1) \times \mathbf{U}(1)$ be the three-dimensional flag manifold. Consider the following reductive decomposition of $\mathbf{u}(3)$

$$\mathbf{u}(3) = \mathbf{h} \oplus \mathbf{m},$$

where $\mathbf{u}(3)$ is the Lie algebra of the unitary group $\mathbf{U}(3)$ and $\mathbf{h}$ and $\mathbf{m}$ are determined by:

$$\mathbf{h} = \left\{ \begin{array}{ccc} i\alpha & 0 & 0 \\ 0 & i\beta & 0 \\ 0 & 0 & i\gamma \end{array} \right\} \cong \mathbf{u}(1) \oplus \mathbf{u}(1) \oplus \mathbf{u}(1) \subset \mathbf{u}(3);$$

$$\mathbf{m} = \left\{ \begin{array}{ccc} 0 & a & b \\ -\bar{a} & 0 & c \\ -\bar{b} & -\bar{c} & 0 \end{array} \right\} \subset \mathbf{u}(3).$$

Identifying any element $X \in \mathbf{m}$ with the triple $(a, b, c)$ we consider the $U(3)$-invariant Riemannian metric on $F_{1,2}$

$$g_{\lambda,\mu,\nu}(X, X) = \lambda|a|^2 + \mu|b|^2 + \nu|c|^2,$$

where $\lambda, \mu, \nu$ are positive real numbers. When $\lambda, \mu, \nu$ vary the set of the metrics $g_{\lambda,\mu,\nu}$ exhaust all $U(3)$-left invariant metrics on $F_{1,2}$.

It is well known that $F_{1,2}$ can be considered as the negative twistor space of $\mathbf{CP}^2$ with the Fubini-Study metric, see [10, 13.80]. It is easily shown that the metrics $h_t$, $t>0$ coincide with the *invariant* metrics $g_{1,\frac{st}{6},1}$ and the complex structures $\mathcal{J}_0$, $\mathcal{J}_1$ and $\mathcal{J}_2$ defined above are the *invariant* complex structures on $F_{1,2}$, acting on the elements of $\mathbf{m}$ by

$$\begin{aligned} \mathcal{J}_0(a,b,c) &= (\;\;ia, \;\;\;ib, \;\;\;ic); \\ \mathcal{J}_1(a,b,c) &= (-ia, -ib, \;\;\;ic); \\ \mathcal{J}_2(a,b,c) &= (\;\;ia, -ib, -ic). \end{aligned}$$

We also observe that the non-integrable almost complex structure $\mathcal{J}_3$ is also invariant and acts by

$$\mathcal{J}_3(a,b,c) = (-ia, \;\;\;ib, -ic).$$

Moreover, for $st=6$ we obtain the *bi-invariant* metric on $F_{1,2}$ and for $st=12$ $(h_t, \mathcal{J}_0)$ is one of the invariant Kähler-Einstein structures on $F_{1,2}$, see [21, 7]. Our main observation concerning $F_{1,2}$ is given by the following

**Theorem 1** *The only (local) orthogonal complex structures compatible with some of the metrics $h_t, t>0$ are the invariant ones.*

*Proof:* Except for the case $\frac{st}{12} = \frac{5+\sqrt{33}}{4}$ the claim follows from Proposition 5. Suppose now that $\frac{st}{12} = \frac{5+\sqrt{33}}{4}$ and let $\mathcal{J}$ be a compatible complex structure, different from $\mathcal{J}_0, \mathcal{J}_1$ and $\mathcal{J}_2$. According to Proposition 5, at any point $\mathcal{J}$ belongs to the family (c) with $\beta_1=\beta_3=0$ or $\beta_2=\beta_3=0$. Suppose for example that in the neighborhood $\mathcal{U}$ of some point $\mathcal{J}$ belongs to the family (c) with $\beta_1=\beta_3=0$. Then, on $\mathcal{U}$, the complex structure $\mathcal{J}$ is compatible with the invariant metric $g_{\frac{6}{st},1,1}$. It is well known (see for example [2]) that there exists an automorphism of $F_{1.2}$ coming from an element of the Weyl group of $\mathbf{SU}(3)$ which switches the metrics $g_{\frac{6}{st},1,1}$ and $g_{1,\frac{6}{st},1}$. But existence of a complex structure compatible with $g_{1,\frac{6}{st},1}$ different from $\mathcal{J}_0, \mathcal{J}_1$ and $\mathcal{J}_2$ contradicts to Proposition 5. □

**Remark 3** Using the algebraic structure of $F_{1,2}$ the authors have recently obtained the same result for *any* invariant metric $g_{\lambda,\mu,\nu}$. A detailed proof and some other applications will appear in a forthcoming paper.

As a consequence of Theorem 1 we obtain

**Corollary 2** *The only orthogonal complex structures compatible with some invariant Einstein metric on the homogeneous space $F_{1,2}$ are the invariant ones.*

*Proof:* Any invariant Einstein metric on $F_{1,2}$ is isometric to either $h_{\frac{6}{s}}$ or $h_{\frac{12}{s}}$ (see for example [7]). □

Another consequence of Theorem 1 is the following

**Corollary 3** *For any Einstein metric of positive scalar curvature on $F_{1,2}$ which is Kähler with respect to some compatible complex structure there are exactly three commuting orthogonal complex structures.*

*Proof:* For any such Kähler structure, say $(h, \mathcal{J})$, on $F_{1,2}$ the complex structure $\mathcal{J}$ has positive first Chern class, hence, since $F_{1,2}$ is spin-manifold, $(F_{1,2}, \mathcal{J})$ is a Fano manifold of index at least 2. From the classification of 3-dimensional Fano manifolds [26, 31] it follows that $\mathcal{J}$ is biholomorphic to one of the invariant complex structures $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2$. Moreover, according to uniqueness of Kähler-Einstein metrics modulo biholomorphisms [9, 30, 29], we can assume that $(h, \mathcal{J})$ is equivalent to $(h_t, \mathcal{J}_0)$ with $st = 12$ and our claim follows from Theorem 1. □ (*End of Example 2.*)

To complete our discussion we consider the case (C) when the spectrum of $W^+$ is not degenerate. We then have $Q \neq 0$, so Lemmas 2, 4 and 5 give

**Proposition 6** *Let $(M, g)$ be an oriented self-dual Einstein Riemannian 4-manifold. Suppose that the spectrum of positive Weyl tensor $W^+$ is non degenerate. After replacing $M$ by a finite covering if necessary, denote by $J'$ and $J''$ the globally defined by Proposition 2-*(i) *positive orthogonal almost complex structures on $(M, g)$ and let $\mathcal{J}'_1$, $\mathcal{J}'_2$ and $\mathcal{J}''_1$, $\mathcal{J}''_2$ be the corresponding lifts to the negative twistor space $\mathcal{Z}$. Then:*

(i) *if the scalar curvature $s$ is non zero then $Z_0(\mathcal{Z}, h_t)$ consists of $\pm\mathcal{J}_0$, $\pm\mathcal{J}'_1$, $\pm\mathcal{J}'_2$, $\pm\mathcal{J}''_1$, $\pm\mathcal{J}''_1$ and $\pm\mathcal{J}_3$;*

(ii) *if the scalar curvature is zero then $Z_0(\mathcal{Z}, h_t)$ has 7 connected components given by $\pm\mathcal{J}_0$, $\pm\mathcal{J}'_1$, $\pm\mathcal{J}'_2$, $\pm\mathcal{J}''_1$, $\pm\mathcal{J}''_2$, $\pm\mathcal{J}_3$ and the $\mathbf{CP}^1$-bundle of the lifts of the negative almost Hermitian structures on $(M, g)$.*

Concerning the twistor spaces of *compact* Einstein self-dual four manifolds, we describe these of them that admit orthogonal complex structure different from $\mathbf{J}$ with the following

**Theorem 2** *Let $(M, g)$ be a compact, self-dual, Einstein Riemannian 4-manifold. Then there exists in the neighborhood of some (or equivalently of each) point of the negative twistor space $\mathcal{Z}$ a positive $h_t$-orthogonal complex structuredifferent from $\mathbf{J}$ for* any $t > 0$ *if and only if $(M, g)$ is isometrically equivalent to one of the following:*

(i) *$\mathbf{CP}^2$ with the Fubini-Study metric;*

(ii) *an oriented compact flat 4-manifold;*

(iii) *a K3 surface or its quotients by groups $\mathbf{Z}_2$ or $\mathbf{Z}_2 \times \mathbf{Z}_2$, endowed with the opposite orientation and a Calabi-Yau Ricci-flatmetric;*

(iv) *a compact quotient of the (complex) hyperbolic space $\mathbf{CH}^2$ with the Bergman metric.*

*Proof:* If $(M, g)$ is $\mathbf{CP}^2$ with the Fubini-Study metric then there are (locally and globally) exactly three orthogonal complex structures associated with any of the metrics $h_t, t > 0$, as already proved in Theorem 1. According to Proposition 5 and Lemma 2, the same is true for the compact quotients of the complex hyperbolic space with the Bergman metric when $\frac{st}{12} \neq \frac{5-\sqrt{33}}{4}$; for $\frac{st}{12} = \frac{5-\sqrt{33}}{4}$ we do not know if there exist some positive orthogonal complex structures different from $\mathcal{J}_0$, $\mathcal{J}_1$ and $\mathcal{J}_2$. When $(M, g)$ is a flat torus or a K3-surface endowed with a Calabi-Yau metric, then according to Lemmas 2 and 3 besides $\mathcal{J}_0$, the other $h_t$-orthogonal complex structures of $\mathcal{Z}$ are the lifts of the positive and negative orthogonal complex structures of $(M, g)$. Since any globally defined orthogonal complex structure on the flat torus and a K3 surface is Kähler in the case of the flat torus there are exactly two $S^2$-families of globally defined complex structures, commuting with but different from $\mathcal{J}_0$ (corresponding to two hyper-Kähler families of $(M, g)$) while in the case of a K3 surface there is exactly one $S^2$-family of globally defined complex structures, commuting with but different from $\mathcal{J}_0$.

Suppose now that on a simply-connected open set $\mathcal{U}$ of $(\mathcal{Z}, h_t)$ there is a positive orthogonal complex structure $\mathcal{J}$ which differs from $\mathcal{J}_0$. Denote by $U$ the projection of $\mathcal{U}$ by $\pi_-$. We claim that either thescalar curvature of $(U,g)$ vanishes or the spectrum of $W^+$ of $(U,g)$ isdegenerate. Indeed, suppose that $s \neq 0$ and the spectrum of $W^+$ is notdegenerate. Then Proposition 6 implies that $\mathcal{J}$ is one of thestructures $\mathcal{J}_1'$, $\mathcal{J}_1''$, $\mathcal{J}_2'$, $\mathcal{J}_2''$. In any case $\mathcal{J}$ is projectable into a positive orthogonal almost complex structure $J$ on $(U,g)$ which coincides with one of the structures $\pm J'$ or $\pm J''$ defined by Proposition 2-(i). The integrability condition for $\mathcal{J}$ implies that $J$ is an integrable structure on $(U,g)$ (see Lemma 2), hence the spectrum of $W^+$ is degenerate by Proposition 3, a contradiction. Since $g$ is an Einstein metric (in particular it is real analytic) we obtain that either $(M,g,J)$ is of zero scalar curvature or the spectrum of $W^+$ is degenerate everywhere on $M$. In the first case a result of Hitchin [22] implies that $(M,g)$ is isometric to one of the manifolds given in (ii) and (iii) of Theorem 2. In the second case, according to Proposition 3, either $(M,g)$ is of constant sectional curvature (i.e. $(M,g)$ is isometric to $S^4$ or to a compact quotient of the real hyperbolic space) or $W^+$ has exactly two distinct eigenvalues at any point of $M$. If $(M,g)$ is of constant non-zero sectional curvature, then it follows from Proposition 4 that $\mathcal{J}$ is projectable to a positive orthogonal almost complex structure $J$ on $(M,g)$ which is Kähler by Lemma 2. This is impossible since for a Kähler surface the positive Weyl tensor vanishes iff the scalar curvature is zero (see for example [18, 16]). If $W^+$ has two distinct eigenvalues, by replacing $M$ by two-fold covering if necessary, we obtain that there is a Hermitian structure $J$ on $(M,g)$, see [16]. According to the classification of the compact self-dual Einstein Hermitian surfaces, obtained by Boyer in [11], $(M,g,J)$ is isometrically and biholomorphically equivalent to one of the following Kähler surfaces: $\mathbf{CP}^2$ with the Fubini-Study metric; a flat complex surface; a compact quotient of the unit ball in $\mathbf{C}^2$ with its Bergman metric, which completes the proof. $\square$

**Remark 4**

1. While $S^4$ and the manifolds appearing in (i), (ii) and (iii) are the only oriented compact self-dual Einstein manifolds of non-negative scalar curvature, see [22, 23, 21], in the case of negative scalar curvature a complete classification is not available and the only known examples are the compact quotients of real and complex hyperbolic spaces. (If we not assume that $M$ is compact some other examples appear, cf. [24].)

2. The proof of Theorem 1 shows that if for *some* $t > 0$ there exists a positive $h_t$-orthogonal complex structure different from $\mathbf{J}$ then $(M,g)$ is either of constant non-zero sectional curvature or isometrically equivalent to one of the Riemannian manifolds described in (i)-(iv) of Theorem 2. In the case when $(M,g)$ is the 4-sphere $S^4$ the metric $h_t$ with $t = \frac{12}{s}$ is the Fubini-Study metric on $\mathbf{CP}^3$, which locally admits infinitely many orthogonal complex structures (see [33]) while for the twistor space of the compact quotients of real hyperbolic space the authors do not know if there are $h_t$-orthogonal complex structures different from $\mathcal{J}_0$. According to Proposition 4 if any, then $\frac{st}{12} = 1 - \sqrt{\frac{3}{2}}$.

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