An estimate for the first eigenvalue of the Dirac operator on compact Riemannian spin manifold admitting parallel one-form

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Abstract

An estimate for the first eigenvalue of the Dirac operator on compact Riemannian spin manifold of positive scalar curvature admitting a parallel 1-form is found. The possible universal covering spaces of the manifolds on which the smallest possible eigenvalue is attained are also listed. Moreover, a complete classification of the compact odd-dimensional manifolds whose universal covering space is $S^{n-1} \times \mathbf{R}$ is given in the limiting case. All such manifolds are diffeomorphic but not necessarily isometric to $S^{n-1} \times S^1$.

Running title: First eigenvalue of the Dirac operator

Keywords. Riemannian manifold, Dirac operator, first eigenvalue of Dirac operator, Killing spinors

MS classification: 53C15; 53C25; 53B35

* The author supported by Contract MM 413/1994 with the Ministry of Science and Education of Bulgaria and by Contract 219/1994 with the University of Sofia "St. Kl. Ohridski".

† The author supported by Contract MM 413/1994 with the Ministry of Science and Education of Bulgaria and by Contract 219/1994 with the University of Sofia "St. Kl. Ohridski".

‡ The author supported by Contract MM 423/1994 with the Ministry of Science and Education of Bulgaria and by Contract 219/1994 with the University of Sofia "St. Kl. Ohridski".

1 Introduction

Let M be a compact Riemannian spin n-manifold of positive scalar curvature s. In [7] T.Friedrich proved that any eigenvalue λ of the Dirac operator D satisfies the inequality

$$(1.1) \qquad \lambda^2 \geq \frac{n}{n-1}\frac{\inf_M s}{4}$$

and in case of equality the corresponding eigenspinor is Killing spinor (see the definition below). Killing spinors are automatically eigenspinors for D of smallest possible eigenvalue.

Afterwards O.Hijazi showed in [14] that if the manifold M admits a parallel k-form, $k \neq 0, n$, then on M do not exist Killing spinors . Hence, the estimate (1.1) is not sharp on such manifolds. Indeed, better estimates have been proved for Kähler spin manifolds by K.-D.Kirchberg in [19, 21] (see also [20, 25, 10, 23] for the limiting case) and for quaternionic Kähler manifolds by W.Kramer, U.Semmelmann and G.Weingart [22] (see also [16, 17, 24]).

In this note we improve (1.1) in the case of manifolds admitting a parallel 1-form. Namely, we prove the following

Theorem 1.1 *Let M be a compact n-dimensional Riemannian spin manifold of positive scalar curvature s admitting a parallel 1-form. Then any eigenvalue of the Dirac operator satisfies the inequality*

$$(1.2) \qquad \lambda^2 \geq \frac{n-1}{n-2}\frac{\inf_M s}{4}.$$

A spinor field ψ on Riemannian spin manifold is called Killing spinor with Killing constant α if for all tangent vectors X the equation $\nabla_X\psi = \alpha X.\psi$ holds. Here $X.\psi$ denotes the Clifford product of X and ψ. If M carries a Killing spinor then it is Einstein with scalar curvature $s = 4n(n-1)\alpha^2$. There are three distinct possibilities: α can be zero and in this case ψ is a parallel spinor field; α can be purely imaginary, then M is noncompact and ψ is called an imaginary Killing spinor, and finally, α can be real, then M is compact (if complete) and ψ is called a real Killing spinor.

In [18] N.Hitchin showed that the manifolds with parallel spinors can be characterized by their holonomy group (see also [8, 26]). Manifolds with imaginary Killing spinors have been classified by H.Baum in [1, 2].

Many results on real Killing spinors had been known (see [9, 15, 13, 11]) until C.Bär [4] gave a description of all complete simply connected Riemannian manifolds carrying real Killing spinors. Using his results, in Theorem 3.1 we list the universal covering spaces of the manifolds for which the limiting case in Theorem 1.1 is attained. We also prove that on all compact odd-dimensional oriented Riemannian manifolds admitting a parallel 1-form whose universal covering space is $S^{n-1} \times \mathbf{R}$ the limiting case is attained. We show that all such manifolds are diffeomorphic but not necessarily isometric to $S^{n-1} \times S^1$. Thus we obtain all odd-dimensional limiting manifolds with exception of dimension 7, where other limiting manifolds also exist.

Finally, we would like to thank the referee for pointing out the incorrect formulation of Theorem 3.1 in the first version of this paper.

2 Proof of Theorem 1.1

Let (M, g) be a Riemannian spin manifold of dimension n with a spinor bundle S. Let TM be the tangent bundle of M, T^*M – the cotangent bundle and $< \cdot, \cdot >$ – the standard Hermitian inner product on S. Denote by ∇ the covariant derivative of the Levi-Civita connection on both TM and S.

Let $\mu : TM \otimes S \longrightarrow S$ be the Clifford multiplication. The Dirac operator $D : \Gamma(S) \longrightarrow \Gamma(S)$ is defined by $D = \mu \circ \nabla$. If $e_1, e_2, ..., e_n$ is a local orthonormal frame on M then

$$D\psi = \sum_{i=1}^{n} e_i \nabla_{e_i}\psi, \qquad \psi \in \Gamma(S).$$

It is well known that the Dirac operator coincides with its formal adjoint and satisfies the Lichnerovicz formula:

$$D^2 = \Delta + \frac{s}{4}$$

Let θ be a parallel 1-form on M and ξ be the dual vector field. Since the length of ξ is constant we can assume that it is of unit length. Hence, we can consider ξ as a section of the endomorphisms of S with the property $\xi^2 = -id$. We denote by $S_{\pm 1}$ the eigenspaces of ξ corresponding to the eigenvalues $\pm\sqrt{-1}$ respectively. Note that $S_{\pm 1}$ are not the half spinor bundles when the dimension of M is even. We have the orthogonal splitting $S = S_1 \oplus S_{-1}$ and $S_{\pm 1}$ are parallel, i.e. ∇:$\Gamma(S_r) \longrightarrow \Gamma(T^*M \otimes S_r)$, $r = \pm 1$. If $X \in TM$, $X \perp \xi$, then for any $\psi \in S_r$ we have $X\psi \in S_{-r}$, $r = \pm 1$.

Let $\pi_r : TM \otimes S_r \longrightarrow Ker\mu|_{TM\otimes S_r}$, $r = \pm 1$, be the orthogonal projections. Then we have

$$\pi_r(X \otimes \psi) = X \otimes \psi + \sum_{i=1}^{n} e_i \otimes p(e_i)p(X)\psi + \frac{1}{n-1}\sum_{i=1}^{n} e_i \otimes \bar{p}(e_i)\bar{p}(X)\psi,$$

where p is the projection on ξ, $\bar{p}$ is the projection on the $(n-1)$-dimensional orthogonal complement of ξ, $X \in \Gamma(TM)$, $\psi \in \Gamma(S_r)$, $r = \pm 1$.

We consider the following differential operators (the twistor operators) $P_r = \pi_r \circ \nabla$, $r = \pm 1$. We have

$$P_r\psi = \sum_{i=1}^{n} e_i \otimes \nabla_{e_i}\psi + \sum_{i,j=1}^{n} e_i \otimes p(e_i)p(e_j)\nabla_{e_j}\psi + \frac{1}{n-1}\sum_{i,j=1}^{n} e_i \otimes \bar{p}(e_i)\bar{p}(e_j)\nabla_{e_j}\psi, \qquad \psi \in \Gamma(S_r). \tag{2.3}$$

The expressions for P_1 and P_{-1} differ only by the domain of ψ, so from now on we will omit the index when writing P_r.

Let the local orthonormal frame $e_1, e_2, ..., e_n$ be such that $e_n = \xi$. Then the equation (2.3) is equivalent to

$$P\psi = \nabla'\psi + \frac{1}{n-1}\sum_{i=1}^{n-1} e_i \otimes e_i D'\psi, \qquad \psi \in \Gamma(S_r), \tag{2.4}$$

where

$$\nabla'\psi = \sum_{i=1}^{n-1} e_i \otimes \nabla_{e_i}\psi, \qquad D'\psi = \mu \circ \nabla'\psi$$

Now, let M be a compact manifold with positive scalar curvature s. Squaring (2.4), integrating over M and using the Lichnerovicz formula we obtain

$$(D^2\psi, \psi) = \frac{n-1}{n-2}\|P\psi\|^2 + \|\nabla_\xi\psi\|^2 + \frac{n-1}{n-2}\int_M < \frac{s}{4}\psi, \psi > dV, \quad \psi \in \Gamma(S_r), \tag{2.5}$$

where $(\psi, \varphi) = \int_M < \psi, \varphi > dV$ for $\psi, \varphi \in \Gamma(S)$.
Let λ be an eigenvalue of D and let ψ be an eigenspinor corresponding to the eigenvalue λ^2 of D^2. Since D^2 preserves the splitting $S = S_1 \oplus S_{-1}$ we can choose $\psi \in \Gamma(S_r)$ for $r = 1$ or $r = -1$. Hence, from (2.5) we obtain the estimate (1.2), which completes the proof of Theorem 1.1.

3 The limiting case

Let we have equality in (1.2). Then it follows from (2.5) that the eigenspinor $\psi \in \Gamma(S_r)$ of D^2 satisfies the equations

(3.6) $$P\psi = 0, \qquad \nabla_\xi \psi = 0.$$

Hence, $D\psi = D'\psi \in \Gamma(S_{-r})$ and it follows from (2.4) that

(3.7) $$\nabla' \psi + \frac{1}{n-1} \sum_{i=1}^{n-1} e_i \otimes e_i D\psi = 0.$$

The spinor field $D\psi \in \Gamma(S_{-r})$ is also an eigenspinor of D^2 for the eigenvalue λ^2 and hence satisfies (3.6) and (3.7) too. Let $\varphi = \psi + \frac{1}{\lambda} D\psi$. Then it follows from (3.6) and (3.7) that

(3.8) $$\nabla_\xi \varphi = 0, \qquad \nabla_X \varphi = \nabla'_X \varphi = -\frac{\lambda}{n-1} X.\varphi, \qquad X \perp \xi.$$

Let $\tilde{M}$ be the universal covering space of M. Then $\tilde{M}$ carries a parallel vector field $\tilde{\xi}$ of unit length and a spinor field $\tilde{\varphi}$ satisfying (3.8). In particular, $\tilde{M}$ is isometric to the product $\tilde{F} \times \mathbf{R}$. Given a spin manifold $M = F \times \mathbf{R}$ then there exists a natural spin structure on F induced by that of M and conversely: if F is a spin manifold then the product $M = F \times \mathbf{R}$ carries a natural spin structure, which induces the given spin structure on F (see [2, 3]). Applying this fact to $\tilde{M}$ we get from (3.8) that $\tilde{\varphi}$ is a real Killing spinor on $\tilde{F}$ (it is clear that ∇' is the Levi-Civita connection on $\tilde{F}$).

Thus we obtain that if on an n-dimensional spin manifold M admitting a parallel 1-form the limiting case of the inequality (1.2) occurs, then the universal covering space $\tilde{M}$ of M is isometric to the product $\tilde{F} \times \mathbf{R}$, where $\tilde{F}$ is a simply connected compact Einstein manifold of dimension $(n-1)$ admitting a real Killing spinor.

Using the classification of all simply connected complete Riemannian manifolds carrying real Killing spinor given by C.Bär in [4] (Theorems $1, 2', 3', 4', 5'$) we obtain the following

Theorem 3.1 *Let M be a compact n-dimensional Riemannian spin manifold of positive scalar curvature s admitting a parallel 1-form and let the first eigenvalue λ of the Dirac operator on M satisfy the equality $\lambda^2 = \frac{n-1}{n-2} \frac{\inf_M s}{4}$. Then the universal covering space $\tilde{M}$ of M is isometric to the Riemannian product $\tilde{F} \times \mathbf{R}$, where up to multiplication of the metric with constant $\tilde{F}$ is one of the following:*

1) $\tilde{F} = S^{n-1}$, $n \geq 3$.

2) $\tilde{F}$ is nearly Kähler non-Kähler manifold, $n = 7$.

3) $\tilde{F}$ is an Einstein-Sasaki manifold, $n = 4m + 2$, $m \geq 1$ odd.

4) $\tilde{F}$ is an Einstein-Sasaki manifold, but does not carry a Sasaki-3-structure, $n = 4m$, $m \geq 2$.

5) $\tilde{F}$ carries a Sasaki-3-structure, $n = 4m$, $m \geq 2$.

6) $\tilde{F}$ carries a nearly parallel vector cross product (in the sense of A.Gray – see [12]), but not a Sasaki structure, $n = 8$.

Remark: Let $\tilde{F}$ be simply connected complete manifold carrying real Killing spinor ψ. Then it is clear that ψ gives rise to spinor field $\tilde{\varphi}$ on $\tilde{F} \times \mathbf{R}$ satisfying (3.8). Similarly, ψ giv

Now we restrict our attention on the case $\tilde{F} = S^{n-1}$ with n odd. In this case the converse of Theorem 3.1 is true. More precisely, we have

Theorem 3.2 *Let M be a compact odd-dimensional oriented Riemannian manifold admitting a parallel 1-form whose universal covering space is $S^{n-1} \times \mathbf{R}$. Then* *a) The manifold M is diffeomorphic but not necessarily isometric to $S^{n-1}\times S^1$ with the standard product metric.*

b) On the manifold M the limiting case in (1.2) is attained.

Proof. Let Γ be the fundamental group of M. Then Γ acts on $S^{n-1}\times\mathbf{R}$ from the left and this action is free and properly discontinuous. Since Γ must preserve the metric and the orientation on $S^{n-1}\times\mathbf{R}$ and the tangent vector field $\frac{d}{dt}$ of $\mathbf{R}$, we easily obtain that Γ is isomorphic to $\mathbf{Z}$, $\Gamma=\{(a^k,k\alpha):k\in\mathbf{Z}\}$, $\alpha\neq 0$. Here the action of $(a,\alpha)\in SO(n)\times\mathbf{R}$ on $S^{n-1}\times\mathbf{R}$ is given by

$$S^{n-1}\times\mathbf{R}\ni(x,t)\to(ax,t+\alpha).$$

Let $\pi:S^{n-1}\times\mathbf{R}\to M$ be the projection. Denote by S^1_α the circle with length α. Let $\pi_1:S^{n-1}\times\mathbf{R}\to S^1_\alpha$ be the map defined by $\pi_1(x,t)=\frac{\alpha}{2\pi}e^{\frac{2\pi\sqrt{-1}t}{\alpha}}$. There exists a map $\pi_2:M\to S^1_\alpha$ such that $\pi_2\circ\pi=\pi_1$ and since π and π_1 are Riemannian submersions, π_2 is Riemannian submersion too. Hence $\pi_2:M\to S^1_\alpha$ is locally trivial fibre bundle. It is easy to see that its fibers are isometric to S^{n-1} and are totally geodesic submanifolds of M. Thus $\pi_2:M\to S^1_\alpha$ is locally trivial fibre bundle with standard fibre S^{n-1} and structure group contained in $O(n)$. Hence, M is diffeomorphic (but not necessarily isometric) to the trivial bundle $S^{n-1}\times S^1$, which proves a).

To prove b) we first recall some facts about spin structures on quotients of S^{n-1} (see [5]).

The total space of the bundle of oriented orthonormal frames of S^{n-1} is $SO(n)$. The spin structure of S^{n-1} is given by the double covering map $\Theta:Spin(n)\to SO(n)$ and $Spin(n-1)$ acts on the total space $Spin(n)$ as structure group from the right. Spinor fields over S^{n-1} can be regarded as $Spin(n-1)$-equivariant mappings from $Spin(n)$ to the spinor space Σ_{n-1}. The $\pm\frac{1}{2}$-Killing spinors on S^{n-1} are of the form $\psi(g)=\tau^\pm(g^{-1})\psi(1)$, where $\tau^\pm$ are the representations of $Spin(n)$ in Σ_{n-1} whose differential $\tau^\pm_*:spin(n)\to Cl_{n-1}$ is given by

$$\tau^\pm_*(e_ie_j)=e_ie_j,\qquad 1\le i<j\le n-1,$$

$$\tau^\pm_*(e_ie_n)=\pm e_i,\qquad 1\le i\le n-1,$$

(see [6]).

Let Γ be a finite fixed point free subgroup of $SO(n)$. Then the spin structures of the quotient $\Gamma\setminus S^{n-1}$ are in 1-1 correspondence with homomorphisms $\varepsilon:\Gamma\to Spin(n)$ such that $\Theta\circ\varepsilon=id_\Gamma$. The total space of the spin structure is then given by $\varepsilon(\Gamma)\setminus Spin(n)$ and spinor fields over the quotient correspond to $\varepsilon(\Gamma)$-invariant spinor fields over S^{n-1}.

Now let $\tilde M=S^{n-1}\times\mathbf{R}$ with n odd. The spin structure of $\tilde M$ (i.e. the principal fibre bundle with structure group $Spin(n)$ which double covers the bundle of oriented orthonormal frames of $\tilde M$) is reduced to the principal $Spin(n-1)$ bundle $Spin(n)\times\mathbf{R}$. Hence, the spinor bundle of $\tilde M$ is the bundle associated with $Spin(n)\times\mathbf{R}$ with standard fibre $\Sigma_n(=\Sigma_{n-1})$ and the spinor fields over $\tilde M$ are $Spin(n-1)$-equivariant mappings from $Spin(n)\times\mathbf{R}$ to Σ_n.

As we have seen, $M=\Gamma\setminus\tilde M$, where Γ is generated by $(a,\alpha)\in SO(n)\times\mathbf{R}$.

Similarly to the above argument, the spin structures of M are of the type $\varepsilon(\Gamma)\setminus Spin(n)\times\mathbf{R}$, where $\varepsilon(\Gamma)$ is generated by $(\tilde a,\alpha)\in Spin(n)\times\mathbf{R}$ such that $\Theta(\tilde a)=a$.

Hence, to prove b) we have to find $\varepsilon(\Gamma)$-invariant spinor field $\psi:Spin(n)\times\mathbf{R}\to\Sigma_n$ such that for every $t\in\mathbf{R}$ $\psi(\cdot,t)$ is Killing spinor on S^{n-1}.

Thus ψ must be of the form $\psi(g,t)=\tau^\pm(g^{-1})\psi(1,t)$ and

$$\psi(1,t+\alpha)=\tau^\pm(\tilde a)\psi(1,t)\qquad\text{for every}\quad t\in\mathbf{R}. \tag{3.9}$$

The action of $\tau^{\pm}(\tilde{a})$ on $\Sigma_{n-1} = \Sigma_n$ is unitary. Let $\varphi_1^{\pm}, ..., \varphi_{2^m}^{\pm}$ $(m = \frac{n-1}{2})$ be a basis of Σ_n consisting of eigenvectors corresponding to the eigenvalues $\lambda_1^{\pm}, ..., \lambda_{2^m}^{\pm}$ of $\tau^{\pm}(\tilde{a})$. Let $\psi_i^{\pm}(1,t) = (c_i^{\pm})^t \varphi_i^{\pm}$, where $(c_i^{\pm})^{\alpha} = \lambda_i^{\pm}$. Obviously, these spinor fields satisfy the condition (3.9) and this completes the proof.

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To appear in Journal of Geometry and Physics