Riemannian manifold in which the skew-symmetric curvature operator has pointwise constant eigenvalues

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Abstract

We study manifolds where the natural skew-symmetric curvature operator has pointwise constant eigenvalues. We give a local classification (up to isometry) of such manifolds in dimension 4. In dimension 3, we describe such manifolds up to a classification of three - dimensional Riemannian manifolds with principal Ricci curvatures $r_1 = r_2 = 0, r_3$- arbitrary. We give examples of such manifolds in all dimensions which do not have constant sectional curvature; these manifolds are not pointwise Osserman manifolds in general.

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1 Introduction

Curvature is fundamental in Riemannian geometry. It is important to find 'nice' conditions on the curvature which determine the Riemannian metric up to isometry. For example, Riemannian symmetric spaces, which were described by E.Cartan, and Riemannian semi-symmetric spaces, which were described by Z.Szabó in [26], are characterized by well known 'nice' curvature conditions.

The Jacobi operator encodes important information concerning the curvature. If the Jacobi operator has constant (resp. pointwise constant) eigenvalues, then the Riemannian manifold M is said to be a globally Osserman (resp. a pointwise Osserman) manifold. If M is a rank one symmetric space or is flat, then the (local) isometry group acts transitively on the unit sphere bundle of M and the eigenvalues of the Jacobi operator are constant. Osserman [23] wondered if the converse was true; if the eigenvalues of the Jacobi operator are globally constant, need the manifold be at least locally a rank one symmetric space or flat. This question has been answered in the affirmative by Chi [5] if the dimension n is odd, if $n \equiv 2 \bmod 4$, or if $n = 4$; the case $n \equiv 0 \bmod 4$ and $n > 4$ is still open. Pointwise Osserman spaces have been studied by many authors [6, 7, 8]. The local and the global questions are quite different; examples of 4-dimensional pointwise Osserman spaces which are not globally Osserman have been given in [8].

If the eigenvalues of the Jacobi operator are constant along every geodesic, M is called an C-space; if the eigenspaces of the Jacobi operator are parallel along every geodesic, M is called a P-space. These manifolds have been investigated in [1, 2, 3, 4]. A local classification of 3-dimensional C-spaces and of 3-dimensional P-spaces has been given in [2]. Locally conformal flat 4-dimensional C-spaces and locally conformal flat 4-dimensional P-spaces have been described recently in [13, 14].

The skew-symmetric curvature operator R also encodes important information about the curvature and we are grateful to Professor G. Stanilov for bringing it to our attention. We refer to [17, 25] where many of the basic properties of this operator are presented. Riemannian manifolds for which R admits a parallel Jordanian basis along every unit circle are called T-spaces; Riemannian manifolds so that the eigenvalues of R are constant along every unit circle are called O-spaces (see citeN-Y for further details). These manifolds have been studied in [11, 12]. A local description of 3-dimensional O-spaces and T-spaces has been also obtained.

In this note we will study manifolds so that R has pointwise constant eigenvalues; we assume $n \geq 3$ hence forth as this condition is automatic if $n = 2$. If M has constant sectional curvature, i.e. is a real space form, then R has globally constant eigenvalues. If $n = 3$, then R has pointwise constant (resp. globally constant) eigenvalues if and only if either two of its principal Ricci curvatures are zero and the third is a smooth function (resp. constant) or it is a real space form. We refer to Remark 2 below for further details. If $n \geq 3$, we will construct Riemannian manifolds which are warped products of an interval with a space form so that R has pointwise constant eigenvalues; these manifolds will not have constant sectional curvature. The main result of this paper is Theorem A. It gives a local classification (up to isometry) of 4-dimensional Riemannian manifolds so that R has pointwise constant eigenvalues. It shows that such a manifold either has constant sectional curvature or is a warped product as described below in Example 1.

Let $(\mathbf{M}, g)$ be a Riemannian manifold. Let ∇ denote the Levi-Civita connection of the metric g. The skew-symmetric curvature operator is defined by $R_{u,v} = [\nabla_u, \nabla_v] - \nabla_{[u,v]}$

for any orthonormal pair (u,v) of tangent vectors at any point $p \in \mathbf{M}$. The curvature operator $R_{u,v}$ does not depend on the oriented orthonormal basis which is chosen for the oriented 2 plane $E^2 = span\{u,v\}$ [17, 25]; if we reverse the orientation, R changes sign.

Definition 1. A Riemannian manifold $(\mathbf{M},g)$ is said to be *of pointwise constant curvature eigenvalues* if the eigenvalues of the curvature operator $R(u,v)$ do not depend on the orthonormal pair (u,v). A Riemannian manifold $(\mathbf{M},g)$ is said to be *of constant curvature eigenvalues* if the eigenvalues of $R(u,v)$ are global constants.

We note that the product of two Riemannian manifolds with pointwise constant curvature eigenvalues will in general not have pointwise constant curvature eigenvalues. The following example is crucial to our classification:

Example 1. Let $n \geq 3$, let $\mathbf{B^1}$ be an open interval on the real line, possibly unbounded, and let $\mathbf{N^{n-1}}$ be real space form with constant sectional curvature $\mathcal{K}$. Let $\mathbf{f}$ be the smooth function on $\mathbf{B}$ defined by $\mathrm{f} := \sqrt{\mathcal{K}x^2 + Cx + D}$ where C and D are constants satisfying the inequality $C^2 - 4\mathcal{K}D \neq 0$. We define the warped product $\mathbf{M^n} := \mathbf{B^1} \times_{\mathbf{f}} \mathbf{N^{n-1}}$. This is an n-dimensional locally conformal flat Riemannian manifold with pointwise constant curvature. The eigenvalue 0 has multiplicity $(n-2)$. The remaining two eigenvalues are:

$$\pm \frac{\sqrt{-1}\{C^2 - 4\mathcal{K}D\}}{4(\mathcal{K}x^2 + Cx + D)^2}. \tag{1.1}$$

If the sectional curvature of N is non-zero, this manifold is *not* Einstein and hence not Osserman.

The main result of this paper is the following theorem:

Theorem A. *Let $(\mathbf{M},g)$ be a 4-dimensional Riemannian manifold of pointwise constant curvature eigenvalues. Then locally almost everywhere (i.e. around the points of an open and everywhere dense set)* $\mathbf{M}$ *is isometric either to a real space form or to a warped product of the form given in Example 1. Conversely, such Riemannian manifolds do have constant curvature eigenvalues.*

An immediate consequence of Theorem A is the following result.

Theorem B. *Every connected 4-dimensional Riemannian manifold of constant curvature eigenvalues is of constant sectional curvature.*

Remark 1. It is easy to verify that, in dimension 3, the space $\mathbf{M^3} = \mathbf{B^1} \times_{\mathbf{f}} \mathbf{N^2}$ constructed in Example 1 has Ricci eigenvalues $(0, 0, 2\frac{C^2-4\mathcal{K}D}{4(\mathcal{K}x^2+Cx+D)^2})$. Moreover, every 3-dimensional Riemannian manifold $(\mathbf{M},g)$ with pointwise constant curvature eigenvalues is locally (almost everywhere) isometric either to a real space form or to a space with principal Ricci eigenvalues of type $(0,0,r)$, where r is a smooth function on $\mathbf{M}$. Here is a sketch of the proof. Let u, v be tangent vectors at a point p in $\mathbf{M}$. Since $n = 3$, we have

$$trace R^2(u,v) = const = -2r^2. \tag{1.2}$$

Let X_1, X_2, X_3 be an orthonormal basis of $T_p\mathbf{M}$ of Ricci eigenvectors. Since $n = 3$ the curvature tensor is completely determined by the Ricci tensor. Then the condition (1.2) is equivalent to

$$K^2(X_1,X_2) = K^2(X_1,X_3) = K^2(X_2,X_3) = r^2,$$

where $K(X_i, X_j)$ is the sectional curvature determined by the 2-plane $span(X_i, X_j)$. This means that the Ricci eigenvalues are either $(2\varepsilon a, 2\varepsilon a, 2\varepsilon a)$ or $(0, 0, 2\varepsilon a)$, where $\varepsilon = \pm 1$. Let Ω be the subset of $\mathbf{M}$ on which the number of distinct Ricci eigenvalues is locally constant. This set is open and dense in $\mathbf{M}$. Now it is clear that our assertion holds in Ω.

Remark 2. The local classification of 3-dimensional Riemannian manifolds with pointwise constant curvature eigenvalues is given up to a description of Riemannian 3-manifolds with principal Ricci curvatures $(0, 0, r(p))$. To the best of our knowledge, this problem is still not completely solved. Nevertheless, we can obtain a local description of Riemannian 3-manifolds with constant curvature eigenvalues. Indeed, it follows from the considerations in Remark 1 that a Riemannian 3-manifold with constant curvature eigenvalues is locally (almost everywhere) isometric either to a real space form or to a Riemannian manifold with principal Ricci curvatures $(0, 0, r = const)$. All Riemannian 3-manifolds satisfying the latter condition are curvature homogeneous but not locally homogeneous and have been described by O.Kowalski in [19] (see also [20, 24]), namely they depend on two arbitrary functions of one variable.

Remark 3. After seeing our paper, Gilkey, Leahy, and Sadofsky [9] generalized Theorems A and B. They showed that if $\mathbf{M}$ is a connected Riemannian manifold of pointwise constant curvature eigenvalues of dimension $n = 5$, $n = 6$, or $n \geq 9$, then $(\mathbf{M}, g)$ is locally isometric either to a manifold of constant sectional curvature or a warped product constructed in Example 1. They inform us their methods, which use algebraic topology, can not be used to derive Theorem A of this paper, that their paper was motivated by Theorem A, and that they have used many of the techniques we have developed. They do not know if Theorem A generalizes to dimensions $n = 7$ or $n = 8$. As noted above, dimension $n = 3$ is quite different.

2 Algebraic considerations

Let ($\mathbf{M}$,g) be a 4-dimensional Riemannian manifold. Let $T_p\mathbf{M}$ be the tangential space at a fixed point p of M. Let $E^2 = span\{x, y\}$ be a 2-dimensional plane of $T_p\mathbf{M}$. We suppose the skew-symmetric curvature operator $R_{E^2} := R_{x,y} : T_p\mathbf{M} \longrightarrow T_p\mathbf{M}$ has constant eigenvalues $\pm\sqrt{-1}a_1$ and $\pm\sqrt{-1}a_2$ on T_pM and classify the resulting algebraic structures. This is a purely algebraic problem. We apply the Vieta formulas to the characteristic polynomial of R_{E^2} to show that:

Proposition 2.1 *A 4-dimensional Riemannian manifold $(\mathbf{M}, g)$ has pointwise constant curvature eigenvalues if and only if the following conditions hold for every orthonormal basis x,y,z,v of $T_p\mathbf{M}$ and for every 2 plane E_2 of $T_p\mathbf{M}$ we have*

$$R^2_{E^2}yx + R^2_{E^2}zv + R^2_{E^2}zx + R^2_{E^2}vy + R^2_{E^2}vx + R^2_{E^2}zy = a_1^2 + a_2^2, \tag{2.3}$$

$$R_{E^2}yxR_{E^2}zv + R_{E^2}zxR_{E^2}vy - R_{E^2}vxR_{E^2}zy = \varepsilon a_1 a_2, \quad \varepsilon = \pm 1, \tag{2.4}$$

where $R_{E^2}xy = g(R_{E^2}x, y)$.

We set

$$H(x, y, z, v) = R_{xyyx}R_{xyzv} - R_{xyzx}R_{yxvy} + R_{xyvx}R_{yxzy},$$

where R_{xyzv} denotes the curvature tensor of type $(0, 4)$ defined by $R_{xyzv} = g(R_{xy}z, v)$. We substitute $E^2 = span\{x, y\}$ and the orthonormal basis $\bar{x} = \cos\alpha x + \sin\alpha x$, $\bar{y} = y$, $\bar{z} = \sin\alpha x - \cos\alpha z$, $\bar{v} = v$ into (2.3) and (2.4) to obtain the following:

Corollary 2.2 *Let a 4-dimensional Riemannian manifold $(\mathbf{M}, g)$ have pointwise constant curvature eigenvalues at a point p of M. Let $\{x, y, z, v\}$ be any orthonormal basis for T_pM and let $H(x, y, z, v) := R_{xyyx}R_{xyzv} - R_{xyzx}R_{yxvy} + R_{xyvx}R_{yxzy}$. Then we have:*

$$(2.5)(R_{xyyx} + R_{zyyz})R_{yxzy} + R_{xvyx}R_{xvyz} + R_{zvyz}R_{zvyx} + R_{yvxy}R_{yvzy} - R_{xzyx}R_{zxyz} = 0,$$

$$H(x, y, z, v) = H(z, v, x, y) = H(x, z, v, y) = H(x, v, y, z). \tag{2.6}$$

Let $\{x_i\}$ be an orthonormal basis of $T_p\mathbf{M}$. Let Latin indices i, j, k, l be distinct integers ranging between 1 and 4. Define:

$$R_{ijkl} := R(x_i, x_j, x_k, x_l),\ L^k_{ij} := R(x_i, x_k, x_k, x_j),\ K_{ij} := R(x_i, x_j, x_j, x_i).$$

Proposition 2.3 *A 4-dimensional Riemannian manifold $(\mathbf{M}, g)$ has pointwise constant curvature eigenvalues $\pm\sqrt{-1}a_1$ and $\pm\sqrt{-1}a_2$ at a point p of M if and only if there exists an orthonormal basis $\{x_i\}$ for $T_p\mathbf{M}$ such that $L^k_{ij} = 0$ and so that exactly one of the following five cases holds with $\varepsilon = \pm 1$:*

i) $K_{12} = K_{13} = K_{34} = K_{24} = -2a_1$, $K_{14} = K_{23} = a_1$, $R_{1234} = R_{1342} = \varepsilon a_1$, $R_{1423} = \varepsilon(-2a_1)$, $a_2 = -2a_1$, $a_1 \neq 0$, *and M is pointwise Osserman at p.*

ii) $K_{ij} = 0$, $R_{ijkl} = 0$, *and* $a_1 = a_2 = 0$,

iii) $K_{12} = K_{13} = K_{14} = K_{23} = K_{24} = K_{34} = \varepsilon a_1$, $R_{ijkl} = 0$, $a_2 = 0$, *and* $a_1 \neq 0$,

iv) $K_{12} = K_{13} = -K_{14} = -K_{23} = K_{24} = K_{34} = \varepsilon a_1$, $R_{ijkl} = 0$, $a_2 = 0$, *and* $a_1 \neq 0$,

v) $K_{12} = K_{13} = K_{14} = -K_{23} = -K_{24} = -K_{34} = \varepsilon a_1$, $R_{ijkl} = 0$, $a_2 = 0$, *and* $a_1 \neq 0$.

Proof. Let $\{x_i\}$ be eigenvectors of the Ricci tensor with corresponding Ricci eigenvalues r_i. We then have the symmetry $L^k_{ij} = -L^l_{ij}$. We use this symmetry and equations (2.3) and (2.6) to see that $K^2_{ij} = K^2_{kl}$, and that $K_{ij}R_{ijkl} = K_{kl}R_{klij}$. We use these equations and the first Bianchi identity to see exactly one of the following four possibilities holds:

A) $K_{12} = K_{34}$, $K_{13} = K_{24}$, and $K_{14} = K_{23}$;

B) $K_{12} = K_{34}$, $K_{13} = K_{24}$, $K_{14} = -K_{23} \neq 0$, and $R_{1423} = 0$;

C) $K_{12} = K_{34}$, $K_{13} = -K_{24} \neq 0$, $K_{14} = -K_{23} \neq 0$, and $R_{ijkl} = 0$;

D) $K_{12} = -K_{34} \neq 0$, $K_{13} = -K_{24} \neq 0$, $K_{14} = -K_{23} \neq 0$, and $R_{ijkl} = 0$.

To prove Proposition 2.3 we examine each these cases in detail.

Case A. In this case the Ricci eigenvalues are equal. Hence the Ricci tensor is a scalar multiple of the metric and every orthonormal basis consists of Ricci eigenvectors. Let x_1 be arbitrary. We study the Jacobi operator $\lambda_{x_1} = R(x, x_1)x_1$. Let $\{x_i\}$ be an orthonormal basis for $T_p\mathbf{M}$ of eigenvectors of λ_{x_1}. Then we have $L^1_{23} = L^1_{24} = L^1_{34} = 0$. We use this identity and (2.6) to see

$$K_{12}R_{1234} = K_{13}R_{1342} = K_{14}R_{1423} = \varepsilon a_1 a_2. \tag{2.7}$$

We substitute the values (x_1, x_2, x_3, x_4), (x_1, x_3, x_2, x_4), and (x_1, x_2, x_4, x_3) for (x, y, z, v) in equation (2.5) we see that

$$0 = (K_{12} + K_{14})L^2_{13} = (K_{13} + K_{14})L^3_{12} = (K_{12} + K_{13})L^2_{14}. \tag{2.8}$$

Assume $a_1a_2 \neq 0$. It follows from equation (2.7) that $K_{ij} \neq 0$. The first Bianchi identity and (2.7) imply $\frac{1}{K_{12}} + \frac{1}{K_{13}} + \frac{1}{K_{14}} = 0$. Then, by (2.8), $L^k_{ij} = 0$ so by equation (2.3)

$$K^2_{12} + R^2_{1234} = K^2_{13} + R^2_{1342} = K^2_{14} + R^2_{1423} = a^2_1 + a^2_2.$$

We use this identity, equation (2.7), and the first Bianchi identity to see case i) holds.

Next we suppose that $a_1 a_2 = 0$. Without loss of generality we may assume $a_2 = 0$. We substitute the values (x_2, x_1, x_3, x_4), (x_2, x_1, x_4, x_3), and (x_3, x_1, x_4, x_2) for (x, y, z, v) in equation (2.5). We use the symmetry $L^k_{ij} = -L^l_{ij}$ and the identity $L^1_{23} = L^1_{24} = L^1_{34} = 0$ to see that

$$(2.9)\quad L^2_{14}R_{1423} - L^2_{13}L^3_{12} = 0, \quad L^2_{13}R_{1342} - L^2_{14}L^3_{12} = 0, \quad L^3_{12}R_{1234} - L^2_{13}L^2_{14} = 0.$$

If we suppose $L^2_{13}L^2_{14}L^3_{12} \neq 0$, then equation (2.8) implies $K_{ij} = 0$ which contradicts the assumption $L^2_{13}L^2_{14}L^3_{12} \neq 0$. Thus, at least one of $L^2_{14}, L^3_{12}, L^2_{13}$ is zero. Assume that $L^3_{12} = 0$. Then equation (2.9) implies $L^2_{13}L^2_{14} = 0$. Suppose $L^2_{14} \neq 0$ and $L^2_{13} = 0$. By equation (2.9), $R_{1423} = 0$. We use equation (2.7) and the first Bianchi identity to see

A1) $R_{1234} = -R_{1342} \neq 0$ and $K_{12} = K_{13} = 0$ or A2) $R_{1234} = R_{1342} = 0$.

If case A1) holds, we substitute $E^2 = span\{E_1, E_2\}$, $x = E_1$, $y = E_2$, $z = E_3$, $v = E_4$ and $E^2 = span\{E_1, E_4\}$, $x = E_1$, $y = E_4$, $z = E_3$, $v = E_2$ in (2.3) to see $K^2_{14} = a^2_1$ and $(L_{14})^2 + R^2_{1234} = a^2_1$. This contradicts equation (2.3) if we take $X = \sum_{i=1}^4 \alpha_i E_i$ $Y = \sum_{i=1}^4 \beta_i E_i$ to be arbitrary orthonormal vectors, $E^2 = span\{X, Y\}$ and $x = E_1$, $y = E_2$, $z = E_3$, $v = E_4$. Thus we suppose that A2) holds. If $L^2_{14} \neq 0$, replace the orthonormal basis $\{x_i\}$ by the new basis: orthonormal basis:

$$x_{\bar{1}} = \cos\beta x_1 + \sin\beta x_4, \; x_{\bar{2}} = x_2, \; x_{\bar{3}} = x_3, \; x_{\bar{4}} = -\sin\beta x_1 + \cos\beta x_4$$

where $2\cot 2\beta = (K_{12} - K_{13})/(L^2_{14})$. It is easy to see that the conditions of the case A) hold and $L^{\bar{k}}_{\bar{i}\bar{j}} = 0$. Thus, we may assume $L^k_{ij} = 0$. We now use equation (2.3) to see that we have $K^2_{12} = K^2_{13} = K^2_{14} = a^2_1$ which is equivalent to ii), iii) and iv).

Case B. Since $K_{12} = K_{34}$, $K_{13} = K_{24}$, we have $r_1 = r_4$, $r_2 = r_3$. Then $span\{x_1, x_4\}$ and $span\{x_2, x_3\}$ are eigenspaces of the Ricci tensor corresponding to the eigenvalues r_1 and r_2 respectively. In view of the construction we have made above, we may suppose

$$(2.10)\qquad L^2_{14} = 0.$$

Setting $x = x_1$, $y = x_2$, $z = x_3$, $v = x_4$ and $x = x_1$, $y = x_4$, $z = x_3$, $v = x_2$ into (2.5) and taking into account (2.10) and $L^k_{ij} = -L^l_{ij}$ we obtain

$$(K_{12} - K_{14})L^2_{13} + R_{1234}L^1_{24} - L^1_{23}L^3_{12} = 0,$$

$$(K_{14} + K_{12})L^2_{13} - R_{1234}L^1_{24} + L^3_{12}L^1_{23} = 0.$$

We get from the latter two equalities that $K_{14}L^2_{13} = 0$. Similarly $K_{14}L^3_{12} = 0$, $K_{14}L^1_{24} = 0$, $K_{14}L^1_{34} = 0$. Since $K_{14} \neq 0$, $L^3_{12} = L^2_{13} = L^1_{24} = L^1_{34} = 0$. Using (2.6), we have

$$(2.11)\qquad \varepsilon a_1 a_2 = K_{12}R_{1234} = K_{13}R_{1342} = K_{14}R_{1423} = 0.$$

Let $a_2 = 0$. The first Bianchi identity and (2.11) imply that we have either

B1) $R_{1234} = -R_{1342} \neq 0$, $K_{12} = K_{13} = 0$

or

B2) $R_{1234} = R_{1342} = 0$.

The case B1) can be excluded in a similar way as A1).

In the case B2) we may assume $L^1_{23}=0$. Indeed, if $L^1_{23}\neq 0$ then we take the following orthonormal Ricci eigenbasis: $x_{\bar 1}=x_1, x_{\bar 2}=\cos\alpha x_2+\sin\alpha x_3$, $x_{\bar 3}=-\sin\alpha x_2+\cos\alpha x_3$, $x_{\bar 4}=x_4$, where $2\cot 2\alpha=\frac{K_{12}-K_{13}}{L^1_{23}}$, instead of x_1,x_2,x_3,x_4. Simple calculations show that $R_{\bar i\bar j\bar k\bar l}=L^{\bar k}_{\bar i\bar j}=0$. Then, (2.3) implies

$$K^2_{12}=K^2_{13}=K^2_{14}=a^2_1. \tag{2.12}$$

We claim that if $L^k_{ij}=R_{ijkl}=a_1a_2=0$, then

$$K_{ij}K_{kl}=K_{il}K_{kj} \tag{2.13}$$

The formula (2.13) follows immediately inserting the following orthonormal basis
$x=\frac{1}{\sqrt2}(\sin\alpha x_i+x_j+\cos\alpha x_l)$, $y=\frac{1}{\sqrt2}(\cos\alpha x_i+x_k-\sin\alpha x_l)$,
$z=\frac{1}{\sqrt2}(x_i-\sin\alpha x_j-\cos\alpha x_k)$, $v=\frac{1}{\sqrt2}(-\cos\alpha x_j+\sin\alpha x_k+x_l)$
and $E^2=span\{x,y\}$ into (2.4). The equation (2.12) contradicts (2.13).

We complete the proof by considering cases C) and D). Suppose there exists a permutation (r,s,t,u) of the indices $1,2,3,4$ such that $L^r_{su}=L^r_{tu}=L^s_{tr}=L^s_{ur}=0$. We the orthonormal basis $x=x_r$, $y=\cos\alpha x_t-\sin\alpha x_u$, $z=\cos\alpha x_s+\sin^2\alpha x_t+\sin\alpha\cos\alpha x_u$, and $v=-\sin\alpha x_s+\sin\alpha\cos\alpha x_t+\cos^2\alpha x_u$ and $E^2=span\{x,y\}$ in equation (2.4) to see that

$$(K_{rt}+K_{ru})L^t_{rs}=0. \tag{2.14}$$

Case C. We use equation (2.6) to see that

$$L^3_{12}L^1_{34}=L^2_{14}L^1_{23}=-L^2_{13}L^1_{24}=\varepsilon a_1a_2/2. \tag{2.15}$$

Since $R_{ijkl}=0$, equation (2.5) implies

$$(K_{ij}+K_{kj})L^j_{ik}+L^j_{il}L^j_{kl}-L^i_{kj}L^k_{ij}=0. \tag{2.16}$$

The latter identity and $K_{13}\neq 0, K_{14}\neq 0$ give

$$L^1_{23}=L^1_{24}=L^2_{13}=L^2_{14}=0. \tag{2.17}$$

We use equation (2.15) to see $a_1a_2=0$, $L^1_{34}L^3_{12}=0$. Suppose $a_2=0$. We have, up to a possible ordering, the following subcases:

C1) $L^3_{12}\neq 0$ and $L^1_{34}=0$ or C2) $L^k_{ij}=0$.

Suppose that case C1) holds. We set $i=2$, $j=3$, $k=2$,and $l=4$ in equation (2.16) and use equation (2.17) to see that $(K_{13}+K_{23})L^3_{12}=0$. This implies $K_{13}=K_{14}$ which contradicts equation (2.14). Suppose that case C2) holds. We use equation (2.3) to see that $K^2_{12}=K^2_{13}=K^2_{14}=a^2_1\neq 0$. This contradicts equation (2.13).

Case D. Since $R_{ijkl}=0$, the identities (2.15) and (2.16) hold. Equation (2.16) implies $(K_{12}+K_{13})L^1_{23}=0$, $(K_{12}+K_{14})L^1_{24}=0$, and $(K_{13}+K_{14})L^1_{34}=0$. If $L^1_{23}L^1_{24}L^1_{34}\neq 0$, then $K_{12}=K_{13}=K_{14}=0$ which is impossible. Suppose $L^1_{23}=0$. Then $a_1a_2=0$ by equation (2.15). We suppose $a_2=0$. If we assume that there is $L^k_{ij}\neq 0$, we may suppose without loss of generality that $L^3_{12}\neq 0$. Using equations (2.15) and (2.16), we see that we have $L^k_{ij}=0$ for $\{i,j\}\neq\{1,2\}$. Furthermore, $K_{13}=K_{14}$. This contradicts equation

(2.14). Therefore we have $L_{ij}^k = 0$. Thus $K_{12}^2 = K_{13}^2 = K_{14}^2 = a_1^2$ by equation (2.3) which is equivalent to v).

Conversely, if there exists an orthonormal basis such that one of the conditions i), ii), iii), iv) and v) holds we use equations (2.3) and (2.4) to see that $(\mathbf{M}, g)$ has pointwise constant curvature eigenvalues. Note that we can assume $\varepsilon = 1$. **Q.E.D.**

We can draw the following consequence from this analysis:

Corollary 2.4 *Let $(\mathbf{M}, g)$ be a 4 dimensional Riemannian manifold with pointwise constant curvature eigenvalues $\pm\sqrt{-1}a_1$ and $\pm\sqrt{-1}a_2$ at a point p of M. Let r_i be the Ricci eigenvalues of M. Then exactly one of the following five cases holds; these correspond to the cases of Proposition 2.3:*

i) $r_1 = r_2 = r_3 = r_4 = 3a_1$, $a_1 = const \neq 0$ and $a_2 = -2a_1$.
ii) $r_1 = r_2 = r_3 = r_4 = 0$ and $a_1 = a_2 = 0$.
iii) $r_1 = r_2 = r_3 = r_4 = 3a_1$, $a_1 = const \neq 0$ and $a_2 = 0$.
iv) $r_1 = r_2 = r_3 = r_4 = a_1$, $a_1 = const \neq 0$ and $a_2 = 0$,.
v) $r_1 = 3a_1$, $r_2 = r_3 = r_4 = -a_1 \neq 0$ and $a_2 = 0$.

3 Proof of Theorem A and Theorem B

Let $(\mathbf{M}, g)$ be a 4 dimensional Riemannian manifold with pointwise constant curvature eigenvalues at each point p. Let Ric be the Ricci tensor of M. Let $g(\rho(x), y) = Ric(x, y)$ for x, y in $T_p\mathbf{M}$, and p in $\mathbf{M}$ be the Ricci operator; ρ is a self-adjoint linear operator on $T_p\mathbf{M}$.

We first eliminate case i). We suppose the contrary and argue for a contradiction. Suppose there is a point p of M where $a_1 a_2 \neq 0$; by continuity this condition holds in a neighborhood. Since a_1 is constant, M is global Osserman; the eigenvalues of the Jacobi operator are constant. The work of Chi [5] shows that any global Osserman manifold in dimension 4 is locally a rank one symmetric space or is flat. Consequently this eigenvalue structure is impossible. We can also eliminate this case using results of Tricerri and Vanhecke [27]. Introduce a Hermitian almost complex structure by defining $Jx_1 = -x_4$, $Jx_2 = -x_3$, $Jx_3 = x_2$, and $Jx_4 = x_1$; then $J^2 = -1$. Let $\pi_1(X, Y, Z) := g(Y, Z)X - g(X, Z)Y$ and $\pi_2(X, Y, Z) = g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ$. The curvature tensor then takes the form $R = -2a_1\pi_1 + a_1\pi_2$; this contradicts ([27], see p.388).

We therefore may assume the rank of R is at most 2.

We use the theory of self-adjoint operators to see that if r_i are the Ricci eigenvalues then by $r_1 \leq r_2 \leq r_3 \leq r_4$ we can define continuous functions on $\mathbf{M}$. By the same reasons $-a_1^2$ (and hence a_1) is continuous too. Let Ω is the subset of all the points of $\mathbf{M}$ around which the multiplicities of eigenvalues of R_{E^2} do not change. This set is open and everywhere dense in $\mathbf{M}$. Let $p \in \Omega$ and U be its neighbourhood such that either a_1 is identically zero on U or a_1 never vanishes on U. (If $(\mathbf{M}, g)$ is connected and it has constant curvature eigenvalues then we can take $\Omega = \mathbf{M} = U$.) Then Corollary 2.4 implies that if we have in a single point of (U, g) one of possibilities ii), iii), iv) and v) then we have the same possibility in the remaining points of (U, g). The possibilities ii), iii), iv) and v) in Corollary 2.4 are connected in a unique way with possibilities ii), iii), iv) and v) in Proposition 2.3.

If ii) holds in every point of U then $R = 0$. If we have iii) everywhere on U then $R = 3a_1\pi_1$.

If we assume that iv) holds around each point of U then we can find a point $p_0 \in U$ and a local frame field X_i around p_0 for which the conditions iv) of Proposition 2.3 holds. (Such point and frame we can choose in following way: First we choose an orthonormal field X_1, defined on an open subset V of U, then we remove from V all the points in which the multiplicities of eigenvalues of Jacobi operator λ_{X_1} change. Further (eventually decreasing V) we choose a local orthonormal frame field $\{X_i\}$ of Jacobi eigenfields for λ_{X_1} on V. So we have $L^1_{23} = L^1_{24} = L^1_{34} = 0$. We can suppose (eventually decreasing V) that each of the functions L^2_{13}, L^2_{14} and L^3_{12} is identically zero or never vanish on U. At last we choose $p_0 \in V$ and deal with the fields X_i in the same way described in Proposition 2.3 to obtain a frame field $\{X_i\}$ for which the conditions iv) holds.)

Below we use the second Bianchi identity. Let $\omega^k_{ij} = g(\nabla_{X_i}X_j, X_k)$ and $R_{ijkl;h} = (\nabla_{X_h}R)(X_i,X_j,X_k,X_l)$ for $i,j,k,l \in \{1,2,3,4\}$. We have

$$R_{ijji;h} = X_h \circ K_{ij}; \qquad R_{ijjk;h} = -\omega^k_{hi}(K_{kj} - K_{ij}); \qquad R_{ijkl;h} = 0. \tag{3.18}$$

$$\omega^i_{kk}(K_{kj} - K_{ij}) + \omega^i_{jj}(K_{kj} - K_{ik}) - X_i \circ K_{kj} = 0, \tag{3.19}$$

$$\omega^k_{li}(K_{kj} - K_{ij}) - \omega^k_{il}(K_{kj} - K_{lj}) = 0. \tag{3.20}$$

We recall that $a_1 = const$. Let $\mathcal{F}_1 := span\{X_1, X_4\}$ and $\mathcal{F}_2 := span\{X_2, X_3\}$. We set $i = 1$, $j = 4$, and $k = 2$ in equation (3.19) to see $\omega^1_{22} = 0$. We set $i = 2$, $j = 3$, $k = 1$, and $l = 4$ in equation (3.20) to see $\omega^2_{41} = 0$. Since the the equations defining case iv) are invariant under interchanging the indices 1 and 4 or interchanging the indices 2 and 3, we have that

$$\begin{aligned}
&\omega^j_{ii} = 0 \quad \text{for} \quad j \in \{1,4\},\ i \in \{2,3\} \text{ and } j \in \{2,3\},\ i \in \{1,4\},\\
&\omega^k_{ij} = 0 \quad \text{for} \quad k \in \{1,4\},\ \{i,j\} = \{2,3\} \text{ and } k \in \{2,3\},\ \{i,j\} = \{1,4\}.
\end{aligned}$$

Hence $\mathcal{F}_1$ and $\mathcal{F}_2$ are autoparallel. This implies that (V,g) is locally isometric to a Riemannian product $\mathbf{M_1} \times \mathbf{M_2}$ where $\mathbf{M_1}$ and $\mathbf{M_2}$ are of dimension 2. By Corollary 2.4, each of $\mathbf{M_1}$ and $\mathbf{M_2}$ has constant sectional curvature εa_1. However, the product $\mathbf{M_1} \times \mathbf{M_2}$ does not have pointwise constant curvature eigenvalues and thus case iv) is not possible on U. Note that this proves Theorem B.

Let v) hold in every point of U, $p \in U$. Around p we choose a local orthonormal frame field $\{X_i\}$ of eigenfields for Ricci tensor Ric. Such a frame field exists since the multiplicities of r_i do not change on U. For this frame field are valid conditions D) and v) from Proposition 2.3.

Taking $i,j,k \in \{2,3,4\}$, we use equation (3.19) to see that

$$X_i \circ a_1 = 0, \quad i = 2,3,4. \tag{3.21}$$

Setting $k = 1, \quad i,j \in \{2,3,4\}$ into equation (3.19), use equation (3.21) to see

$$\omega^i_{11} = 0, \quad i = 2,3,4. \tag{3.22}$$

Let $i = 1$ and let $j,k \in \{2,3,4\}$. Then $2a_1(\omega^1_{kk} + \omega^1_{jj}) - X_1 \circ a_1 = 0$ by equation (3.19) so

$$\omega^1_{kk} = \frac{X_1 \circ a_1}{4a_1}, \quad k = 2,3,4. \tag{3.23}$$

Taking $l = 1, \quad i,j,k \in \{2,3,4\}$, we calculate by means of (3.20)

$$\omega^1_{ik} = 0, \quad i,k = 2,3,4. \tag{3.24}$$

Let $\mathcal{F}^{\perp} := span\{X_1\}$ and let $\mathcal{F} = span\{X_2, X_3, X_4\}$. We set $\eta = \omega_{22}^1 X_1$. We use equations (3.22), (3.23) and (3.24) to see that we may apply a result of Hiepko ([10], p.210-211) to conclude that (U, g) is locally isometric to a warped product $\mathbf{B^1} \times_{\mathbf{f}} \mathbf{N^3}$ with 1-dimensional base $\mathbf{B^1}$ and 3- dimensional leaf $\mathbf{N^3}$. Let ρ_i, $i = 2, 3, 4$ be a smooth Ricci eigenvalue functions of $\mathbf{N^3}$ and let Y_2, Y_3, Y_4 be a smooth orthonormal frame field on $\mathbf{N^3}$ satisfying the conditions $\rho(Y_i) = \rho_i Y_i, i = 2, 3, 4$. Let Y_1 be a smooth unit field on $\mathbf{B^1}$. Denoting the lift of a vector field (or of a function) on $\mathbf{B^1}$ (or on $\mathbf{N^4}$) to $\mathbf{B^1} \times_{\mathbf{f}} \mathbf{N^3}$ by $\sim$, it follows from the formulas for warped product (see [21], p.204-210) that

$$E_1 = \tilde{Y}_1, \quad E_i = \frac{\tilde{Y}_i}{\tilde{\mathrm{f}}}, \quad i = 2, 3, 4$$

are smooth orthonormal Ricci eigenfields on $\mathbf{B^1} \times_{\mathbf{f}} \mathbf{N^3}$ corresponding to the smooth Ricci eigenfunctions r_1, r_2, r_3, r_4 of $\mathbf{B^1} \times_{\mathbf{f}} \mathbf{N^3}$ given by

$$r_1 = -3\frac{E_1^2(\tilde{\mathrm{f}})}{\tilde{\mathrm{f}}}, \quad r_i = \frac{\tilde{\rho}_i}{\tilde{\mathrm{f}}^2} - 2\left(\frac{E_1(\tilde{\mathrm{f}})}{\tilde{\mathrm{f}}}\right)^2 - \frac{E_1^2(\tilde{\mathrm{f}})}{\tilde{\mathrm{f}}}, \quad i = 2, 3, 4. \tag{3.25}$$

We know from Corollary 2.4 that

$$-\frac{1}{3}r_1 = r_2 = r_3 = r_4 = a_1 \neq 0. \tag{3.26}$$

We use equations (3.25) and (3.26) to see that $\rho_1 = \rho_2 = \rho_3$. This implies that $\mathbf{N^3}$ is an Einstein manifold. Since $\mathbf{N^3}$ is 3-dimensional, it has constant sectional curvature $\mathcal{K}$. Hence $\rho_i = 2K, \quad i = 2, 3, 4$ and from (3.26) and equation (3.25) shows that

$$\frac{E_1^2(\tilde{\mathrm{f}})}{\tilde{\mathrm{f}}} = 2\left(\frac{\mathcal{K}}{\tilde{\mathrm{f}}^2} - \left(\frac{E_1(\tilde{\mathrm{f}})}{\tilde{\mathrm{f}}}\right)^2\right) - \frac{E_1^2(\tilde{\mathrm{f}})}{\tilde{\mathrm{f}}}, \quad \text{i.e.} E_1^2(\tilde{\mathrm{f}}^2) = 2\mathcal{K}.$$

If we denote by x the local coordinate on $\mathbf{B^1}$ the latter equality implies $\mathrm{f}^2 = \mathcal{K}x^2 + Cx + D$ where C, D are constants. The condition $r_1 = -3a_1 \neq 0$ implies $C^2 - 4\mathcal{K}D \neq 0$. Conversely, it is easy to check that $\mathbf{B^1} \times_{\mathbf{f}} \mathbf{N^3}$ is a space of pointwise constant curvature eigenvalues.

This completes the proof of Theorem A and its consequence Theorem B. **Q.E.D.**

Remark 4. The coefficients of the characteristic polynomial of the skew-symmetric curvature operator of a Riemannian manifold with pointwise constant curvature eigenvalues are also pointwise constant by the Vieta formulas. The Riemannian 4-manifolds satisfying the weaker condition that only one of the two coefficients of the characteristic polynomial of the skew-symmetric curvature operator is pointwise constant are considered in [25, 17, 15, 16].

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