CURVATURE PROPERTIES OF TWISTOR SPACES OF QUATERNIONIC KÄHLER MANIFOLDS [1]

Dedicated to Professor Helmut Karzel on the occasion of his 70th birthday.

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We present a study of natural almost Hermitian structures on twistor spaces of quaternionic Kähler manifolds. This is used to supply $(4n+2)$-dimensional examples $(n>1)$ of symplectic non-Kähler manifolds. Studying their curvature properties we give a negative answer to the questions raised by D.Blair-S.Ianus and A.Gray, respectively, of whether a compact almost Kähler manifold with Hermitian Ricci tensor or whose curvature tensor belongs to the class $AH2$ is Kähler.

1 INTRODUCTION

The problem of searching for examples of compact symplectic non-Kähler manifolds has received much attention recently. After the first example due to W.Thurston [21], much is known about the topology of them. Using the ideas of A.Reznikov [18] we show that some compact locally homogeneous manifolds appearing as a twistor spaces of certain quaternionic Kähler locally symmetric spaces supply new examples of compact symplectic non-Kähler manifolds. Moreover, we obtain explicit expressions of the curvature of some natural family of Riemannian metrics. This relates to another open questions of Almost Hermitian geometry. We emphasize on two of them. The first one was raised by D.Blair and S.Ianus [4] and the second one by A.Gray [13].

For a given compact symplectic manifold $\mathbf{M}$, D.E.Blair and S.Ianus [4] considered the integrals $\int_{\mathbf{M}} s\, dV_g$ and $\int_{\mathbf{M}} (s-s^*)\, dV_g$, s (resp. s^*) being the scalar (resp. $*$-scalar) curvature, as functionals on the set of metrics associated with the symplectic structure and proved that the critical points of these functionals are the associated almost Kähler metrics for which the Ricci tensor is Hermitian with respect to the corresponding almost complex structure. Since the Kähler metrics satisfy this condition, Blair and Ianus raised the question of whether a compact almost Kähler manifold with Hermitian Ricci tensor is Kähler.

In [13] A.Gray introduced three basic classes $AH1, AH2, AH3$ of almost Hermitian mani-

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folds which curvature tensor resembles that of a Kähler manifold. In [12] S.Goldberg has proved that every compact almost Kähler manifold belonging to the class $AH1$ is Kähler. In [13] A.Gray raised the question whether the same is true for the class $AH2$.

Negative answers to both of the questions are first given in dimension 6: to the first question by J.Davidov and O.Muškarov [8] and to the second one by J.Davidov, G.Grantcharov and O.Muškarov [9].

In this note, we give negative answers to the both questions in dimension $4n+2, n \geq 2$ by considering the twistor space of a compact quaternionic Kähler manifold.

The twistor space of quaternionic Kähler manifold $\mathbf{M}$ is a 2-sphere bundle Z over $\mathbf{M}$ whose fibre at any point $p \in \mathbf{M}$ consists of all complex structures on the tangent space $T_p\mathbf{M}$ at p which are compatible with the given quaternionic structure. The $(4n+2)$-dimensional manifold Z admits a natural 1-parameter family of Riemannian metrics $h_c, c > 0$ such that the projection $\pi : Z \to \mathbf{M}$ is a Riemannian submersion. These metrics are compatible with the two almost complex structures J_1 and J_2 defined in the 4-dimensional case by M.Atiyah, N.Hitchin and I.Singer [2] and J.Eells and S.Salamon [10], respectively. S.Salamon [19] and Berard-Bergery [3] have proved that for particular choice of the parameter c the manifold (Z, h_c, J_1) is Kähler-Einstein if the scalar curvature s of $\mathbf{M}$ is positive.

In this paper we study the almost Hermitian geometry of $(Z, J_i, h_c), i = 1, 2$. We determine the types of $(Z, J_i, h_c), i = 1, 2$, according to Gray-Hervella [14] classification of almost Hermitian manifolds (Theorem 3.1) ; the case $n = 1$ is treated by Muškarov [16]. We give an explicit formula of the Riemannian curvature tensor of h_c in terms of the curvature of $\mathbf{M}$ and study its properties in details (Proposition 3.2). As a consequence we obtain our main result:

THEOREM 1.1 *Let $\mathbf{M}^{4n}$ be a quaternionic Kähler manifold with scalar curvature s and $(Z, J_i, h_c), i = 1, 2$ its twistor space endowed with the almost hermitian structures $(J_i, g_c), i = 1, 2$. Then*

i) The manifolds $(Z, J_i, h_c), i = 1, 2$ belong always to the class $AH2$ and have Hermitian Ricci tensor and Hermitian $*$-Ricci tensor.

ii) If $s < 0$ then the manifold $(Z, J_2, h_c), c^2 = -\frac{4}{s}(n+2)$, is almost Kähler (in particular symplectic).

iii) Suppose moreover $\mathbf{M}^{4n}$ is a compact Riemannian locally symmetric space of negative scalar curvature, which is not locally Hermitian symmetric. Then the twistor space Z of $\mathbf{M}$ does not admit any Kähler structure.

REMARKS

1. Let $\mathbf{M}^{4n}$ be a compact quaternionic Kähler locally symmetric space of negative scalar curvature. Then Theorem 1.1 supplies $4n+2$- dimensional examples giving negative answers to the both questions mentioned above.

2. The condition iii) of Theorem 1.1 supplies new examples of $(4n+2)$-dimensional compact symplectic manifolds which admit complex structure but do not admit any Kähler structure.

3. In [22] C.Wood has introduced the notions of harmonic and stable harmonic almost Hermitian structures in terms of variation of section of general metric twistor bundle. Using his results (Theorem 3.1 of [22]) and condition i) of Theorem 1.1 we see that $(J_i, h_c), i = 1, 2,$ are always harmonic structures on Z.

We would like to thank A.Gray who suggested to us the non-existence of Kähler structures and S.Salamon for his interest and comments on the present note.

2 PRELIMINARIES

The setup in this section is similar to that of [20, 5].

Let $\mathbf{H}$ be the quaternions and identify $\mathbf{H^n} = \mathbf{R^{4n}}$ by the following linear isomorphism

$$\mathbf{H^n} \ni x = x_0 + x_1 i + x_2 j + x_3 k \xrightarrow{h} (x_0, x_1, x_2, x_3)^t \in \mathbf{R^{4n}},$$

where $(x_0, x_1, x_2, x_3)^t$ denotes the column-vector. Let $\mathbf{H}$ act on the right. This defines an antihomomorphism

$$\lambda : \{\text{unit quaternions}\} \longrightarrow SO(4n),$$

where $SO(4n)$ acts on the left. Denote the image by $Sp(1)$ and let $I_0 = \lambda(i), J_0 = \lambda(j), K_0 = \lambda(k)$. The Lie algebra of $Sp(1)$ is $sp(1) = span\{I_0, J_0, K_0\}$. Define $Sp(n) = \{A \in SO(4n) : AB = BA$ for all $B \in Sp(1)\}$. The Lie algebra of $Sp(n)$ is $sp(n) = \{A \in so(4n) : AB = BA$ for all $B \in sp(1)\}$. Let $Sp(n)Sp(1)$ be the product of the two groups in $SO(4n)$. Abstractly, $Sp(n)Sp(1) = Sp(n) \times Sp(1)/\mathbf{Z}_2$. The Lie algebra of the group $Sp(n)Sp(1)$ is isomorphic to $sp(n) \oplus sp(1)$.

We recall that a smooth oriented $4n$-dimensional ($n \geq 2$) Riemannian manifold M is said to be quaternionic Kähler if its holonomy group lies in $Sp(n)Sp(1) \subset SO(4n)$. Note that the case $n = 1$ is meaningless because $Sp(1)Sp(1) = SO(4)$. Thus, M is quaternionic Kähler iff its bundle $SO(M)$ of oriented orthonormal frames can be reduced to a principal $Sp(n)Sp(1)$-bundle $P(M)$, so that the Levi-Civita connection 1-form on $SO(M)$ drops down to $P(M)$. In other words, the restriction of the connection 1-form on $P(M)$ is really $sp(n) \oplus sp(1)$-valued.

Now we give some notations. Let $<,>$ be the canonical inner product in $\mathbf{R^{4n}}$. We denote by A^* (resp. $B(\xi)$) the fundamental vector field (resp. the standard horizontal vector field) on

$P(M)$ corresponding to $A \in sp(n) \oplus sp(1)$ (resp. $\xi \in \mathbf{R^{4n}}$). Let ω, Ω, θ be the connection 1-form, the curvature 2-form and the canonical 1-form for the Levi-Civita connection on $P(M)$ respectively ([15]). Let ∇ be the covariant differentiation of the Levi-Civita connection and R be the curvature tensor on M, i.e. $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ and $R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

There is a well known result of Alekseevskii [1] that quaternionic Kähler manifold is Einstein and its Riemannian curvature is of the form $R = \frac{s}{\tilde{s}}\tilde{R} + R'$, where $\tilde{R}$ is the curvature of the quaternionic projective space $\mathbf{HP^n}$ ($\mathbf{HP^n}$ is diffeomorphic to the symmetric space $Sp(n+1)/Sp(n) \times Sp(1)$), $\tilde{s} = 16n(n+2)$ is the scalar curvature of $\mathbf{HP^n}$, s is the scalar curvature of M and $R' \in Sym^2(sp(n))$ (c.f. also [7]).

We shall denote the corresponding splitting of the $sp(n) \oplus sp(1)$-valued curvature 2-form Ω on $P(M)$ by $\Omega = \frac{s}{\tilde{s}}\tilde{\Omega} + \Omega'$, where $\tilde{\Omega}$ is the $sp(n) \oplus sp(1)$-valued curvature 2-form of $\mathbf{HP^n}$ and Ω' is a $sp(n)$-valued form. There is a splitting

$$\tilde{\Omega} = \tilde{\Omega}'' + \tilde{\Omega}''' \tag{2.1}$$

where $\tilde{\Omega}''$ and $\tilde{\Omega}'''$ are the $sp(n)$ and $sp(1)$ components of $\tilde{\Omega}$ respectively. We shall need the explicit formula for $\tilde{\Omega}'''$. We have

$$\tilde{\Omega}''' = \tilde{\Omega}_1''' I_0 + \tilde{\Omega}_2''' J_0 + \tilde{\Omega}_3''' K_0 \tag{2.2}$$

where $\tilde{\Omega}_i''', i = 1,2,3$, are 2-forms. If $\xi, \eta \in \mathbf{R^{4n}}$, then the 2-forms $\tilde{\Omega}_i''', i = 1,2,3$, are given by

$$\begin{aligned} \tilde{\Omega}_1'''(B(\xi), B(\eta)) &= -2 < I_0\xi, \eta > \\ \tilde{\Omega}_2'''(B(\xi), B(\eta)) &= -2 < J_0\xi, \eta > \\ \tilde{\Omega}_3'''(B(\xi), B(\eta)) &= -2 < K_0\xi, \eta > \end{aligned} \tag{2.3}$$

Now let (M,g) be a $4n$-dimensional quaternionic Kähler manifold and $\pi : P(M) \longrightarrow M$ be the natural projection. For each $u \in P(M)$ we consider the linear isomorphism $j(u)$ on $T_{\pi(u)}M$ defined by $j(u) = uJ_0u^{-1}$.

It is easy to see that $j(u)^2 = -id$ and $g(j(u)X, j(u)Y) = g(X,Y)$ for all $X, Y \in T_{\pi(u)}M$, i.e. $j(u)$ is an orthogonal complex structure at $\pi(u)$. For each point $p \in M$ we define $Z_p(M) = \{j(u) : u \in P(M), \pi(u) = p\}$. In other words, $Z_p(M)$ is the space of all orthogonal complex structures in the tangent space T_pM which are compatible with the quaternionic Kähler structure.

We put $Z = \bigcup_{p \in M} Z_p(M)$.

Let $H = Sp(n)Sp(1) \bigcap U(2n)$. There is a bijective correspondence between the symmetric space $Sp(n)Sp(1)/H = Sp(1)/U(1) = \mathbf{CP^1} = \mathbf{S^2}$ and $Z_p(M)$ for every $p \in M$. So we can consider Z as the associated fibre bundle of $P(M)$ with standard fibre $Sp(n)Sp(1)/H = \mathbf{CP^1}$. Hence, $P(M)$ is a principal fibre bundle over Z with structure group H and projection j. If $\pi_1 : Z \longrightarrow M$ is the projection, we have that $\pi_1 \circ j = \pi$.

Now we consider the symmetric space $Sp(n)Sp(1)/H$. We have the following Cartan decomposition of $sp(n) \oplus sp(1)$:

$$sp(n) \oplus sp(1) = h \oplus m,$$

where $h = \{A \in sp(n) \oplus sp(1) : AJ_0 = J_0A\} = (sp(n) \oplus sp(1)) \bigcap u(2n)$ is the Lie algebra of H and $m = \{A \in sp(n) \oplus sp(1) : AJ_0 = -J_0A\}$. It is clear that m is generated by I_0 and K_0, i.e.

$$m = span\{I_0, K_0\}. \tag{2.4}$$

Hence, if $A \in m$ then $J_0A \in m$.

Let $(,)$ be the inner product in $gl(4n, \mathbf{R})$ defined by

$$(A, B) = trace(AB^t) = \sum_{i=1}^{4n} < Ae_i, Be_i > \text{for} A, B \in gl(4n, \mathbf{R}).$$

It is clear that $sp(n) \perp sp(1)$ and I_0, J_0, K_0 form an orthogonal basis of $sp(1)$ with $(I_0, I_0) = (J_0, J_0) = (K_0, K_0) = 4n$. Hence, $h \perp m$.

Let $u \in P(M)$ and Q_u is the horizontal subspace of the tangent space $T_uP(M)$ induced by the Levi-Civita connection on M ([15]). The vertical space is $h^*_u \oplus m^*_u$, where $h^*_u = \{A^*_u : A \in h\}, m^*_u = \{A^*_u : A \in m\}$. Hence, $T_uP(M) = h^*_u \oplus m^*_u \oplus Q_u$.

For each $u \in P(M)$ we put $V_{j(u)} = j_{*u}(h^*_u \oplus m^*_u), H_{j(u)} = j_{*u}Q_u$. Thus we obtain the vertical and horizontal distributions V and H on Z. Since $P(M)$ is a principal fibre bundle over Z with structure group H we have $Kerj_{*u} = h^*_u$. Hence $V_{j(u)} = j_{*u}m^*_u$ and $j_{*u|m^*_u \oplus Q_u} : m^*_u \oplus Q_u \longrightarrow T_{j(u)}Z$ is an isomorphism.

We define almost complex structures J_1 and J_2 on Z by

$$\begin{aligned} J_1 j_{*u} A^* &= j_{*u}(J_0A)^* \\ J_2 j_{*u} A^* &= -j_{*u}(J_0A)^* \\ J_i j_{*u} B(\xi) &= j_{*u} B(J_0\xi), \qquad i = 1, 2, \end{aligned} \tag{2.5}$$

for $u \in P(M), A \in m, \xi \in \mathbf{R^{4n}}$.

For twistor bundles of 4-manifolds the almost complex structure J_1 is introduced in [2] and the almost complex structure J_2 is introduced in [10] .

For every $c > 0$ a Riemannian metric h_c on Z is defined by

$$h_c(j_{*u}A^*, j_{*u}B^*) = c^2(A, B);$$

$$h_c(j_{*u}A^*, j_{*u}B(\xi)) = 0 \tag{2.6}$$

$$h_c(j_{*u}B(\xi), j_{*u}B(\eta)) =< \xi, \eta >,$$

for $u \in P(M), A, B \in m, \xi, \eta \in \mathbf{R^{4n}}$.
It is clear that $(J_i, h_c), i = 1, 2$, determine two families of almost Hermitian structures on Z.

Let Φ_i be the Kähler form of $(J_i, h_c), i = 1, 2$, and let $F_i(X, Y, Z) = (D_X\Phi_i)(Y, Z) = h_c((D_X J_i)Y, Z)$, where D is the covariant derivative of the Levi-Civita connection of h_c. We denote by K the curvature tensor of h_c.

Since the calculations made in [20] for the metric twistor space over Riemannian manifold can be performed in our case, using (2.5) and (2.6) we have :

$$\begin{gathered}
F_i(j_{*u}A^*, j_{*u}B^*, j_{*u}C^*) = 0, \qquad i = 1, 2\\
F_i(j_{*u}A^*, j_{*u}B^*, j_{*u}B(\xi)) = 0, \qquad i = 1, 2\\
F_i(j_{*u}A^*, j_{*u}B(\xi), j_{*u}B(\eta)) = \frac{c^2}{2}(A, \Omega(B(J_0\xi), B(\eta))(u))\\
+\frac{c^2}{2}(A, \Omega(B(\xi), B(J_0\eta))(u)) + 2 < J_0\xi, \eta >, \qquad i = 1, 2\\
F_i(j_{*u}B(\xi), j_{*u}A^*, j_{*u}B^*) = 0, \qquad i = 1, 2\\
F_1(j_{*u}B(\xi), j_{*u}A^*, j_{*u}B(\eta)) =\\
-\frac{c^2}{2}(A, J_0\Omega(B(\xi), B(\eta))(u)) + \frac{c^2}{2}(A, \Omega(B(\xi), B(J_0\eta))(u))\\
F_2(j_{*u}B(\xi), j_{*u}A^*, j_{*u}B(\eta)) =\\
\frac{c^2}{2}(A, J_0\Omega(B(\xi), B(\eta))(u)) + \frac{c^2}{2}(A, \Omega(B(\xi), B(J_0\eta))(u))\\
F_i(j_{*u}B(\xi), j_{*u}B(\eta), j_{*u}B(\zeta)) = 0, \qquad i = 1, 2
\end{gathered} \tag{2.7}$$

For the curvature tensor K we have

$$K(j_{*u}A^*, j_{*u}B^*, j_{*u}C^*, j_{*u}D^*) = -c^2([A, B], [C, D])$$

$$K(j_{*u}A^*, j_{*u}B^*, j_{*u}C^*, j_{*u}B(\xi)) = 0$$

$$K(j_{*u}A^*, j_{*u}B(\xi), j_{*u}B^*, j_{*u}B(\eta)) =$$

$$-\frac{c^4}{4}(B, \Omega(B(\xi), B(e_i))(u))(A, \Omega(B(\eta), B(e_i))(u))+$$

$$(2.8)\qquad +\frac{c^2}{2}([A,B],\Omega(B(\xi),B(\eta))(u))$$

$$K(j_{*u}B(\xi),j_{*u}B(\eta),j_{*u}B(\zeta),j_{*u}A^*)=\frac{c^2}{2}(A,B(\zeta)_u(\Omega(B(\xi),B(\eta))))$$

$$K(j_{*u}B(\xi),j_{*u}B(\eta),j_{*u}B(\zeta),j_{*u}B(\tau))=R(u(\xi),u(\eta),u(\zeta),u(\tau))-$$

$$-\frac{c^4}{4}\{(\Omega_m(B(\eta),B(\zeta))(u),\Omega_m(B(\xi),B(\tau))(u))-$$

$$-(\Omega_m(B(\xi),B(\zeta))(u),\Omega_m(B(\eta),B(\tau))(u))-$$

$$-2(\Omega_m(B(\xi),B(\eta))(u),\Omega_m(B(\zeta),B(\tau))(u))\}$$

for $u\in P(M), A,B,C,D\in m,\xi,\eta,\zeta,\tau\in\mathbf{R^{4n}}$, where Ω_m denotes the m-component of Ω.

Since in the case of quaternionic Kähler manifold we know more about the curvature, from (2.1), (2.2), (2.3), (2.4), (2.7) we obtain

$$F_i(j_{*u}A^*,j_{*u}B^*,j_{*u}C^*)=0,\qquad i=1,2$$

$$F_i(j_{*u}A^*,j_{*u}B^*,j_{*u}B(\xi))=0,\qquad i=1,2$$

$$F_i(j_{*u}A^*,j_{*u}B(\xi),j_{*u}B(\eta))=(8n\frac{s}{\tilde{s}}c^2-2)<AJ_0\xi,\eta>,\qquad i=1,2$$

$$(2.9)\qquad F_i(j_{*u}B(\xi),j_{*u}A^*,j_{*u}B^*)=0,\qquad i=1,2$$

$$F_1(j_{*u}B(\xi),j_{*u}A^*,j_{*u}B(\eta))=0$$

$$F_2(j_{*u}B(\xi),j_{*u}A^*,j_{*u}B(\eta))=8n\frac{s}{\tilde{s}}c^2<AJ_0\xi,\eta>$$

$$F_i(j_{*u}B(\xi),j_{*u}B(\eta),j_{*u}B(\zeta))=0,\qquad i=1,2$$

Using also (2.8) we get

$$K(j_{*u}A^*,j_{*u}B^*,j_{*u}C^*,j_{*u}D^*)=-\frac{c^2}{n}(J_0A,B)(J_0C,D)$$

$$K(j_{*u}A^*,j_{*u}B^*,j_{*u}C^*,j_{*u}B(\xi))=0$$

$$K(j_{*u}A^*,j_{*u}B(\xi),j_{*u}B^*,j_{*u}B(\eta))=$$

$$-4n\frac{s^2}{\tilde{s}^2}c^4(A,B)<\xi,\eta>+(4n\frac{s^2}{\tilde{s}^2}c^4-2\frac{s}{\tilde{s}}c^2)(J_0A,B)<J_0\xi,\eta>$$

$$(2.10)\qquad K(j_{*u}B(\xi),j_{*u}B(\eta),j_{*u}B(\zeta),j_{*u}A^*)=0$$

$$K(j_{*u}B(\xi),j_{*u}B(\eta),j_{*u}B(\zeta),j_{*u}B(\tau))=R(u(\xi),u(\eta),u(\zeta),u(\tau))$$

$$-4n\frac{s^2}{\tilde{s}^2}c^2\{<I_0\eta,\zeta><I_0\xi,\tau>+<K_0\eta,\zeta><K_0\xi,\tau>-$$

$$-<I_0\xi,\zeta><I_0\eta,\tau>-<K_0\xi,\zeta><K_0\eta,\tau>-$$

$$-2<I_0\xi,\eta><I_0\zeta,\tau>-2<K_0\xi,\eta><K_0\zeta,\tau>\}$$

for $u\in P(M), A,B,C,D\in m,\xi,\eta,\zeta,\tau\in\mathbf{R^{4n}}$.

3 CURVATURE PROPERTIES OF (Z, h_c, J_i)

Let (M, g, J) be a $2n$-dimensional almost Hermitian manifold and let

$$F(X, Y, Z) = g((\nabla_X J)Y, Z),$$

where ∇ is the covariant differentiation of the Levi-Civita connection on M. We recall the definition of some classes according to the Gray-Hervella classification [14] in terms of the notations we use:

(M, g, J) is Kähler iff $F = 0$,

(M, g, J) is Hermitian iff $F(X, Y, Z) - F(JX, JY, Z) = 0$,

(M, g, J) is semi-Kähler iff $\delta F = 0$, where $\delta F(X) = \sum_{i=1}^{2n} F(E_i, E_i, X)$ for arbitrary orthonormal basis $E_1, \ldots, E_{2n}$,

(M, g, J) is quasi-Kähler iff $F(X, Y, Z) + F(JX, JY, Z) = 0$

(M, g, J) is nearly Kähler iff $F(X, Y, Z) + F(Y, X, Z) = 0$

(M, g, J) is almost Kähler iff $F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) = 0$
where X, Y, Z are arbitrary vectors at an arbitrary point $p \in M$.

Now we can state the classification results for $(Z, J_i, h_c), i = 1, 2$, as almost Hermitian manifolds (the first statement of the following theorem is proved in [17] for certain choice of the metric h_c).

THEOREM 3.1 *Let M be a $4n$-dimensional quaternionic Kähler manifold with scalar curvature s. Then*

1) For every $c > 0$ (Z, J_1, h_c) is Hermitian semi-Kähler manifold.

- (Z, J_1, h_c) is Kähler manifold iff $sc^2 = 4(n + 2)$.

2) For every $c > 0$ (Z, J_2, h_c) is quasi-Kähler non-Kähler manifold.

- (Z, J_2, h_c) is nearly Kähler iff $sc^2 = 2(n + 2)$.

- (Z, J_2, h_c) is almost Kähler iff $sc^2 = -4(n + 2)$.

Proof: It follows from (2.9) by straightforward computations.

The $*$-Ricci tensor of a $2n$-dimensional almost Hermitian manifold is defined by $\rho^*(X, Y) = \sum_{i=1}^{2n} R(E_i, X, JY, JE_i)$, where R denotes the curvature of the metric g, $E_1, ..., E_{2n}$ is an orthonormal basis at an arbitrary point p and X, Y are tangent vectors at p. By the Bianchi identity we easily get the formula $\rho^*(X, Y) = -\frac{1}{2} \sum_{i=1}^{2n} R(X, JY, E_i, JE_i)$. If the $*$-Ricci tensor is scalar multiple of the metric then the manifold is said to be $*$-Einstein.

We recall also the three basic classes $AH1, AH2, AH3$ of almost Hermitian manifolds introduced by A.Gray [13]. They are defined by the following curvature identities:

$AH1$: $\quad R(X,Y,Z,W) = R(X,Y,JZ,JW)$;

$AH2$: $\quad R(X,Y,Z,W) = R(JX,JY,Z,W) + R(JX,Y,JZ,W) + R(JX,Y,Z,JW)$;

$AH3$: $\quad R(X,Y,Z,W) = R(JX,JY,JZ,JW)$.

It is easy to see that $AH1 \subset AH2 \subset AH3$. The main curvature properties of $(Z, h_c, J_i), i = 1, 2$, are studied in the following

PROPOSITION 3.2 *The manifolds $(Z, J_i, h_c), i = 1, 2$, belong always to $AH2$ and are with Hermitian Ricci tensor and with Hermitian $*$-Ricci tensor:*

a) The manifolds (Z, J_1, h_c) belong to AH1 iff $s = 0$ or $sc^2 = 4(n + 2)$;

b) The manifolds (Z, J_2, h_c) belong to AH1 iff $s = 0$.

c) The manifolds $(Z, J_i, h_c), i = 1, 2$, are Einstein iff $sc^2 = 4(n + 2)$ or $sc^2 = \frac{4(n+2)}{n+1}$;

d) The manifolds (Z, J_1, h_c) are $*$-Einstein iff $sc^2 = 4(n + 2)$ or $sc^2 = -\frac{4(n+2)}{n}$;

e) The manifolds (Z, J_2, h_c) are $*$-Einstein iff

$$sc^2 = \frac{2(n+2)}{n-1}\left[3n - 1 - \sqrt{9n^2 - 10n + 5}\right]$$

or

$$sc^2 = \frac{2(n+2)}{n-1}\left[3n - 1 + \sqrt{9n^2 - 10n + 5}\right].$$

Proof. From (2.10) we obtain the following formulas for the Ricci tensor ρ of (Z, h_c)

$$\begin{aligned} \rho(j_{*u}A^*, j_{*u}B^*) &= \left[16n^2\frac{s^2}{\tilde{s}^2}c^2 + \frac{1}{nc^2}\right] h_c(j_{*u}A^*, j_{*u}B^*); \\ \rho(j_{*u}A^*, j_{*u}B(\xi)) &= 0; \\ \rho(j_{*u}B(\xi), j_{*u}B(\eta)) &= \left[\frac{s}{4n} - 16n\frac{s^2}{\tilde{s}^2}c^2\right] h_c(j_{*u}B(\xi), j_{*u}B(\eta)) \end{aligned} \tag{3.11}$$

Similarly we get for the $*$-Ricci tensors ρ_1^* and ρ_2^* for (Z, h_c, J_1) and (Z, h_c, J_2), respectively:

$$\begin{aligned} \rho_1^*(j_{*u}A^*, j_{*u}B^*) &= \left[-16n^2\frac{s^2}{\tilde{s}^2}c^2 + 8n\frac{s}{\tilde{s}} + \frac{1}{nc^2}\right] h_c(j_{*u}A^*, j_{*u}B^*); \\ \rho_2^*(j_{*u}A^*, j_{*u}B^*) &= \left[16n^2\frac{s^2}{\tilde{s}^2}c^2 - 8n\frac{s}{\tilde{s}} + \frac{1}{nc^2}\right] h_c(j_{*u}A^*, j_{*u}B^*); \\ \rho_i^*(j_{*u}A^*, j_{*u}B(\xi)) &= 0 = \rho_i^*(j_{*u}B(\xi), j_{*u}A^*), \quad i = 1, 2; \\ \rho_1^*(j_{*u}B(\xi), j_{*u}B(\eta)) &= 4(n+1)\frac{s}{\tilde{s}}h_c(j_{*u}B(\xi), j_{*u}B(\eta)); \end{aligned} \tag{3.12}$$

$$\rho_2^*(j_{*u}B(\xi), j_{*u}B(\eta)) = \left[16n\frac{s^2}{\tilde{s}^2}c^2 + (4n-4)\frac{s}{\tilde{s}}\right] h_c(j_{*u}B(\xi), j_{*u}B(\eta)).$$

Now the assertion follows from (2.10), the special form of the curvature of $\mathbf{M}$ and (3.11), (3.12). □

4 PROOF OF THEOREM 1.1

The first two statements of Theorem 1.1 follow from Theorem 3.1 and Proposition 3.2.

To prove iii) we shall use similar arguments as those of [18]. Suppose Z admits a Kähler structure. Consider the projection $\pi : Z \to \mathbf{M}$. Since the curvature of $\mathbf{M}$ is non-positive then by result of J.Eells and J.Sampson [11] there is a harmonic map ψ in the homotopy class of π. According to result of J.Carlson and D.Toledo [6] $rank\psi < dim\mathbf{M}$ at every point since $\mathbf{M}$ is a Riemannian locally symmetric space, but not Hermitian locally symmetric. Using Lerre-Hirsh theorem for the bundle $\pi : Z \to \mathbf{M}$, one could see that $[\psi^* V] \neq 0$ where $V \in H^n(\mathbf{M}, \mathbf{R})$ is the fundamental class. Since $\psi^* V = 0$ this gives a contradiction and the proof of Theorem 1.1 is completed. □

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