EINSTEIN-HERMITIAN SURFACES AND HERMITIAN EINSTEIN-WEYL STRUCTURES IN DIMENSION 4

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Abstract. We show that an Einstein-Weyl structure D on a 4-dimensional conformal manifold $(M,[g])$ preserves some $[g]$-orthogonal complex structure J if and only if the (conformal) Hermitian structure $([g],J)$ is Einstein. In the compact case, we then infer that Einstein-Hermitian complex surfaces are Einstein-Kähler or of Hopf type.

Introduction

Let (M,g,J) be a Hermitian manifold of (real) dimension $n=2m>2$, where g is a Riemannian metric and J a g-orthogonal, integrable, almost- complex structure on M.

Let E be a holomorphic vector bundle of rank r over M. Then, a (fibred) Hermitian metric h on E is said to be Einstein (with respect to g) if the curvature R^∇ of the associated Chern connection ∇ satisfies the following condition, cf. [7] :

$$\text{(E)} \qquad R^\nabla(\Omega)=\frac{u}{r}\,iI_E\,,$$

where: u is some (real) function, I_E denotes the identity of E, $R^\nabla(\Omega)$ is the image of the Kähler form Ω of (g,J) by R^∇, considered as a homomorphism of the bundle Λ^2M of real 2-forms of M into the bundle of skew-hermitian endomorphisms of E.

In case that M is compact, the Hermitian structure (g,J) can be made standard by a conformal change of g (recall that (g,J) is said to be standard if Ω^{m-1} is dd^c-closed, or, equivalently, the Lee form $\theta=J\delta\Omega$ is co-closed) ; then, the function u can be made constant, essentially equal to the degree of E with respect to the polarization Ω, by a conformal change of h, cf. [3]. It is well-known that the Einstein condition (E) is closely related to the stability of E with respect to the (standard) polarization Ω, cf. for example, [2] Ch.6 and enclosed references.

In the case that E is the tangent bundle TM, we can consider the equation (E) for $h=\frac{1}{2}(g-i\Omega)$, the Hermitian metric induced by the pair (g,J) on TM; then, (E) becomes a (highly non-linear) equation for g itself, better expressed as:

$$\text{(E}-\text{H)} \qquad R^\nabla(\Omega)=\frac{u}{m}\,\Omega\,,$$

where ∇ denotes the Chern connection of the Hermitian structure (g,J), while $R^\nabla(\Omega)$ is now considered as a J-invariant, real, 2-form on M. It can be checked that the condition (E-H) is conformally invariant, i.e. is satisfied for any element of the conformal class $[g]$ of g, whenever it is satisfied for g.

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In particular, if M is compact, an Einstein-Hermitian structure (g, J) can be assumed to be standard without loss of generality. Even in this case, there is no guarantee a priori that the (Hermitian) scalar curvature u appearing in (E-H) should be constant, although this fact happens to be true for $n = 4$, as a consequence of Theorem 2 (if u is a *negative* constant, it follows from Theorem 2 that (g, J) is actually Einstein-Kähler; this fact can also be checked directly, cf. [5]).

We obtain another conformally invariant Einstein-like condition by considering Weyl structures with respect to the conformal class $[g]$. Recall that a Weyl structure with respect to $[g]$ is defined as a torsion-free connection acting on sections of TM, preserving the conformal class $[g]$. Then, a Weyl structure is said to be Einstein if the associated (symmetric) Ricci tensor is a multiple of any metric g in $[g]$.

It has been known for a long time that any Hermitian structure (g, J) determines a well-defined Weyl structure D with respect to $[g]$, only depending upon $[g]$ and J, cf. [10] and (1) below. Then, the condition that D be an Einstein -Weyl structure is clearly conformally invariant.

In this paper, we concentrate our attention to the case that the (real) dimension n of M is equal to 4. In this case, the canonical Weyl structure D associated with any Hermitian structure (g, J) also preserves the (integrable) almost-complex structure J.

This property is not satisfied, in general, for n greater than 4, and strong restrictions hold for the existence of Einstein-Weyl structures with respect to $[g]$, preserving some $[g]$-orthogonal complex structure J for $n > 4$, cf. [8].

It could be hoped that the situation be better for $n = 4$, concerning the existence of Hermitian structures whose canonical Weyl structure D satisfies the Einstein condition. Similarly, there has been some hope that Einstein-Hermitian metrics could possibly be found on compact complex surfaces where it is known that no Einstein-Kähler metric exists, cf. [6].

In the present paper, we give a negative answer in both directions. More precisely, we prove the two following facts for (standard) compact Hermitian manifold (M, g, J) of (real) dimension 4:

1. (g, J) is an Einstein-Hermitian structure if and only if the canonical Weyl structure D is Einstein (Theorem 1).
2. This only happens in the two following situations:
   (i) (g, J) is Einstein-Kähler, or
   (ii) the complex surface (M, J) is a Hopf surface and g is locally isometric to the product $\mathbb{R} \times S^3$ (up to homothety) (Theorems 2 and 3).

As already observed, a precise description of Einstein-Weyl structures preserving some orthogonal complex structure for $n > 4$ has been given in [8]. On the other hand, it is still unknown, whether the fact 2 above still holds (mutatis mutandis) for $n > 4$.

Conformally invariant Einstein-like conditions for hermitian complex surfaces.

Let (M, g, J) be a Hermitian manifold of (real) dimension 4, where, as above, g is a Riemannian metric and J an integrable, g-orthogonal almost-complex structure on M.

We denote by:

Ω the *Kähler form*, defined by: $\Omega = g(J.,.)$;

θ the *Lee form*, defined equivalently by: $\theta = J\delta\Omega$ or $d\Omega = \theta\wedge\Omega$ (where δ denotes the codifferential with respect to g);

T the *Lee vector field*, dual of θ with respect to g.

Recall that the Hermitian structure (g, J) is *standard* if and only if the Lee form θ is co-closed : $\delta\theta = 0$.

Besides the Levi-Civita connection D^g of g and the Chern connection ∇ of (g, J), we define the *canonical Weyl structure* D of (g, J) by :

$$D_X Y = D^g_X Y - \frac{1}{2}\,\theta(X)Y - \frac{1}{2}\,\theta(Y)X + \frac{1}{2}\,g(X, Y)T, \tag{1}$$

for any vector fields X and Y.

It is easily checked that D is independent of the choice of g in the conformal class $[g]$ and preserves J. As a matter of fact, D can be characterized as the unique torsion-free connection on the tangent bundle TM preserving both the complex structure J and the conformal structure $[g]$.

The canonical Weyl structure D and the Chern connection ∇ are related by :

$$\nabla = D + \frac{1}{2}\,\theta\otimes I + \frac{1}{2}\,J\theta\otimes J, \tag{2}$$

where I denotes the identity of TM, i.e.:

$$\nabla_X Y = D_X Y + \frac{1}{2}\,\theta(X)Y - \frac{1}{2}\,\theta(JX)JY. \tag{3}$$

The curvature tensors of the connections D^g, ∇ and D are denoted by R^g, R^∇ and R^D respectively (convention: $R^g_{X,Y} = D^g_{[X,Y]} - [D^g_X, D^g_Y]$, etc.).

Notice that R^g and R^∇ are 2-forms with values in the bundle $A(M)$ of skew-symmetric endomorphisms of TM, while R^D takes its values in the direct sum $A(M)\oplus\mathbb{R}.I$.

The curvature tensors R^D and R^∇ are related by:

$$R^D = R^\nabla + \frac{1}{2}\,d(J\theta)\otimes J + \frac{1}{2}d\theta\otimes I. \tag{4}$$

The *Weyl Ricci tensor* Ric^D is defined by:

$$\mathrm{Ric}^D(X, Y) = \sum_{i=1}^{4} g(R^D_{e_i,X}e_i, Y), \tag{5}$$

where $\{e_i\}, i = 1, \ldots, 4$, is any g-orthonormal frame. It can be checked that the tensor Ric^D so defined is a *symmetric* bilinear form (this would not be true any more for $n \neq 4$). On the other hand, Ric^D is conformally invariant, as well as D itself.

Definition 1. *A Weyl structure D is* Einstein *if the trace-free part* Ric_0^D *of* Ric^D *vanishes identically on M.*

The Einstein condition for the canonical Weyl structure D of (g, J) is clearly conformally invariant.

In addition to the Weyl Ricci tensor Ric^D and the Riemannian Ricci tensor Ric^g, defined in a similar way, we shall consider the following Ricci-like 2-forms: $R^\nabla(\Omega)$, $R^g(\Omega)$ and $R^D(\Omega)$, images of the Kähler form Ω by the curvature operators R, R^g, and R^D respectively, i.e.:

$$(R^\nabla(\Omega))(X,Y) = \frac{1}{2}\sum_{i=1}^{4} g\left(R^\nabla_{e_i,Je_i}X, Y\right), \tag{6}$$

and similarly for $R^g(\Omega)$ and $R^D(\Omega)$.

Notice that $R^D(\Omega)$ being actually skew-symmetric, indeed a J-invariant 2-form, can be deduced from (4) and the fact that the differential $d\theta$ of the Lee form of any Hermitian structure (g, J) is primitive, i.e. satisfies:

$$(d\theta, \Omega) = 0. \tag{7}$$

Also notice that the Hermitian Ricci form $R^\nabla(\Omega)$ defined by (6) is distinct, in general, from the 2-form ρ^∇ defined by: $\rho^\nabla(X,Y) = \frac{1}{2}\sum_{i=1}^{4} g\left(R^\nabla_{X,Y}e_i, Je_i\right)$, which is a closed, J-invariant 2-form, and represents the (real) first Chern class of the complex surface (M, J) up to the factor 2π.

Notations. In (6), and henceforth, $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ denotes a J-*adapted*, g-orthonormal frame. In (7), and henceforth, $(.,.)$ denotes the inner product induced by g; for 2-forms ϕ, ψ we adopt the usual convention :

$$(\phi, \psi) = \sum_{i<j} \phi(e_i, e_j).\psi(e_i, e_j) = \frac{1}{2}\sum_{i,j=1}^{4} \phi(e_i, e_j).\psi(e_i, e_j). \tag{8}$$

We denote by k^∇ the symmetric, J-invariant, bilinear form associated to the Hermitian Ricci form $R^\nabla(\Omega)$ by :

$$k^\nabla(X,Y) = (R^\nabla(\Omega))(X, JY). \tag{9}$$

Definition 2. *The Hermitian structure (g, J) is* Einstein *if the trace-free part k_0^∇ of the (symmetric) Hermitian Ricci form k^∇ vanishes identically. Equivalently, (g, J) is Einstein if the condition (E-H) is satisfied.*

As explained in the Introduction, the Einstein condition for Hermitian structures (g, J) is a specialization to the tangent bundle of a general Einstein condition, which makes sense for any Hermitian holomorphic vector bundle over a Hermitian manifold .

It can be checked, cf. [3], that the Einstein condition for the Hermitian structure (g, J) is conformally invariant for any dimension n. In the present situation, when $n = 4$, this fact follows from the following

Theorem 1. *For any Hermitian manifold (M, g, J) of (real) dimension 4, we have the identity:*

$$k_0^\nabla = \mathrm{Ric}_0^D. \tag{10}$$

In particular, (g, J) is an Einstein-Hermitian structure if and only if the canonical Weyl structure D is an Einstein-Weyl structure.

Proof. From (4) and (7), we get:

$$R^D(\Omega) = R^\nabla(\Omega) + \frac{1}{2}\,(d(J\theta), \Omega)\,\Omega. \tag{11}$$

On the other hand, R^D satisfies the first Bianchi identity (since D is a torsion-free), and the identity :

$$\left[R^D_{X,Y}, J\right] = 0, \tag{12}$$

since D preserves J.

We thus get :

$$\begin{aligned}(R^D(\Omega))(X, JY) &= \frac{1}{2}\sum_{i=1}^{4}(R^D_{e_i,Je_i}X, JY)\\ &= -\frac{1}{2}\sum_{i=1}^{4}\Big(\big(R^D_{Je_i,X}e_i, JY\big) + \big(R^D_{X,e_i}Je_i, JY\big)\Big)\\ &= \sum_{i=1}^{4}\big(R^D_{e_i,X}e_i, Y\big)\ ,\end{aligned}$$

hence :

$$\mathrm{Ric}^D(X, Y) = (R^D(\Omega))(X, JY). \tag{13}$$

¿From (11) and (13), we obtain :

$$k^\nabla = \mathrm{Ric}^D - \frac{1}{2}\,(d(J\theta), \Omega)\,g, \tag{14}$$

from which (10) follows immediately. □

A structure theorem for Einstein-Hermitian surfaces

Theorem 2. *Let (M, g, J) be a compact Hermitian manifold of (real) dimension 4. If (g, J) is an Einstein-Hermitian structure and g is standard, then:*

(i) *either (M,g,J) is Kähler-Einstein , or*

(ii) *g is locally isometric (up to homothety) to $\mathbb{R}\times S^3$, the Lee form θ is D^g-parallel, the Weyl structure D is flat, and the complex surface (M, J) is a Hopf surface.*

In view of Theorem 1, Theorem 2 is equivalent to:

Theorem 3. *Let $(M,[g])$ be a compact, oriented, conformal manifold of dimension 4. Let D be an Einstein-Weyl structure with respect to $[g]$. Let g be the associated standard metric. If D preserves some $[g]$-orthogonal complex structure J, then :*
(i) *either D coincides with the Levi-Civita connection D^g of g, and the Hermitian structure (g,J) is Kähler-Einstein, or*
(ii) *D is flat, the metric g is locally isometric (up to homotopy) to $\mathbb{R}\times S^3$, the Lee form θ of the Hermitian structure (g,J) is D^g-parallel, and the complex surface is a Hopf surface.*

Theorems 2-3 come as a consequence of Lemmas 1 and 2 below, and of the following general facts for Einstein-Weyl structures on a compact, 4-dimensional conformal manifold, where g denotes the standard metric, θ the 1-form defined by (1), T the dual vector field with respect to g (cf. [4]):

- (F1) T is a Killing vector field with respect to g (observation due to K.P. Tod [9]).
- (F2) The conformal scalar curvature κ, defined as the trace of Ric^D with respect to g, is constant.
- (F3) The Riemannian Ricci tensor Ric^g of g is equal to :

$$\mathrm{Ric}^g = \frac{\kappa}{4}\,g + \frac{1}{2}\,(|\theta|^2 g - \theta\otimes\theta), \tag{15}$$

cf. [4, I(35)] (WARNING: the 1-form θ in (15) is different by a factor $-\frac{1}{2}$ from the 1-form denoted by θ in [4, I(35)]).

Lemma 1. *For any Hermitian manifold (M,g,J) of (real) dimension 4, we have :*

$$(R^g(\Omega))(X,JX) = \frac{1}{2}\,[\mathrm{Ric}^g(X,X) + \mathrm{Ric}^g(JX,JX)] - \frac{1}{4}\,(|\theta|^2 + 2\delta\theta)|X|^2, \tag{16}$$

for any vector field X.

Proof. Write the Riemannian curvature R^g in the following well-known form:

$$R^g_{X,Y} = \frac{\mathrm{Scal}^g}{12}\,X\wedge Y + \frac{1}{2}\,[\mathrm{Ric}^g_0(X)\wedge Y + X\wedge \mathrm{Ric}^g_0(Y)] + W_{X,Y}, \tag{17}$$

where : Scal^g denotes the Riemannian scalar curvature, i.e. the trace of Ric^g with respect to g; Ric^g_0 the trace-free part of Ric^g; W the Weyl tensor of $[g]$, cf. e.g. [1].

We infer :

$$\begin{aligned}(R^g(\Omega))(X,Y) = \frac{\mathrm{Scal}^g}{12}\,\Omega(X,Y) &+ \frac{1}{2}\,[\mathrm{Ric}^g_0(JX,Y) - \mathrm{Ric}^g_0(X,JY)] \\ &+ (W^+(\Omega))(X,Y),\end{aligned} \tag{18}$$

where W^+ denotes the self-dual Weyl tensor of $[g]$ (recall that the Kähler form Ω is self-dual with respect to the orientation induced by J).

Since $W^+(\Omega)$ is selfdual, its J-invariant part is proportional to Ω itself; more precisely, we have

$$(W^+(\Omega))(X,JX) = \frac{\kappa}{6}|X|^2, \tag{19}$$

as it is easily deduced from (12) (recall that κ is the conformal scalar curvature, appearing in (F2) above).

By (1), the Riemannian scalar curvature Scal^g and the conformal scalar curvature κ are related by :

$$\kappa = \mathrm{Scal}^g - \frac{3}{2}\,(|\theta|^2 + 2\delta\theta). \tag{20}$$

Then, (16) readily follows from (18), (19) and (20). □

Lemma 2. *For any 4-dimensional Hermitian manifold (M, g, J) such that the dual vector field T of the Lee form is a Killing vector field with respect to g, we have :*

$$(R^g(\Omega))(T, JT) = 0. \tag{21}$$

Proof. Let $\{e_i\}$ be a J-adapted, g-orthonormal frame as above, and assume, for convenience, that e_1 and e_3 are D^g-parallel at the point x. Then, at x :

$$\sum_{i=1}^{4} [e_i, Je_i] = JT. \tag{22}$$

Here and in the sequel, we use the well-known identity :

$$D^g_{JX} J = J\, D^g_X J, \tag{23}$$

for any vector field X, easily deduced from (28) below.

We then have:

$$\begin{aligned}
(R^g(\Omega))(T, JT) &= \frac{1}{2} \sum_{i=1}^{4} \left(R^g_{e_i, Je_i} T, JT\right) \\
&= -\frac{1}{2} \sum_{i=1}^{4} \Big(\left(D^g_{e_i} \left(D^g_{Je_i} T\right), JT\right) - \left(D^g_{Je_i} \left(D^g_{e_i} T\right), JT\right) \\
&\qquad\qquad - \left(D^g_{[e_i, Je_i]} T, JT\right) \Big) \\
&= -\sum_{i=1}^{4} \left(D^g_{e_i} \left(D^g_{Je_i} T\right), JT\right) \text{ (by (22) and since T is Killing)} \\
&= -\sum_{i=1}^{4} e_i.\left(D^g_{Je_i} T, JT\right) + \sum_{i=1}^{4} \left(D^g_{Je_i} T, D^g_{e_i}(JT)\right),
\end{aligned} \tag{24}$$

where $e_i.(\;)$ denotes the derivative in the direction of e_i.

Before computing the first term in the latter sum, let us recall the following well-known facts concerning the interplay between the (commuting) actions on (real) 2-forms of the Hodge operator $*$ and the operator J defined by : $(J\psi)(X, Y) = \psi(JX, JY)$. Then, for any (real) 2-form ψ, we have :

$$J\psi = -*\psi + (\psi, \Omega)\,\Omega. \tag{25}$$

In particular, because of (7), the 2-form $d\theta$ satisfies:

$$J(d\theta) = -*d\theta. \tag{26}$$

Then,

$$
\begin{aligned}
(27)\qquad -\sum_{i=1}^{4} e_i.(D^g_{Je_i}T, JT) &= \frac{1}{2}\sum_{i=1}^{4} e_i.\left((J(d\theta))(T, e_i)\right) \text{ (since T is Killing)}\\
&= \frac{1}{2}\,\delta(T\lrcorner * d\theta),
\end{aligned}
$$

where $\lrcorner$ denotes the contraction $T\lrcorner\psi = \psi(T, .)$. Indeed, with respect to the above frame $\{e_i\}$, we have (at x): $\delta\,\alpha = -\sum_{i=1}^{4} e_i.\,\alpha(e_i) + \alpha\,(\sum_{i=1}^{4} D^g_{e_i}e_i) = -\sum_{i=1}^{4} e_i.\,\alpha(e_i) - \frac{1}{2}\alpha(T)$, for any 1-form α; then, apply this identity to $\alpha = T\lrcorner * d\theta$.

In order to evaluate the second term in the last sum in of (22), we need the following expression for the covariant derivative of J with respect to D^g (easily deduced from (1), since J is preserved by D) :

$$
(28)\qquad (D^g_X J)(Y) = \frac{1}{2}\,\theta(JY)X - \frac{1}{2}\,\theta(Y)JX + \frac{1}{2}\,\Omega(X, Y)T + \frac{1}{2}\,g(X, Y)JT.
$$

Specializing by $Y = T$ (the Lee vector field), we get :

$$
(29)\qquad D^g_X(JT) = J(D^g_X T) + \frac{1}{2}\,\theta(JX)T + \frac{1}{2}\,\theta(X)JT - \frac{1}{2}|\theta|^2 JX.
$$

We then infer :

$$
\begin{aligned}
\sum_{i=1}^{4}\left(D^g_{Je_i}T, D^g_{e_i}JT\right) =&\sum_{i=1}^{4}\Big(\left(D^g_{Je_i}T, JD^g_{e_i}T\right) \;\; + \frac{1}{2}\,\theta(Je_i)\left(D^g_{Je_i}T, T\right)\\
&+ \frac{1}{2}\,\theta(e_i)\left(D^g_{Je_i}T, JT\right) - \frac{1}{2}\,|\theta|^2\left(D^g_{Je_i}T, Je_i\right)\Big)\\
=&\sum_{i=1}^{4}\left(D^g_{Je_i}T, JD^g e_i T\right) + \frac{1}{2}\,\left(D^g_T T, T\right) + \frac{1}{2}\,\left(D^g_{JT}T, JT\right) + \frac{1}{2}\,|\theta|^2\delta\theta\\
=&\sum_{i=1}^{4}\left(D^g_{Je_i}T, JD^g_{e_i}T\right),
\end{aligned}
$$

since T is Killing.

Then,

$$
\begin{aligned}
\sum_{i=1}^{4}\left(D^g_{Je_i}T, JD^g_{e_i}T\right) &= -\sum_{i,j=1}^{4}\left(D^g_{Je_i}T, e_j\right)\left(D^g_{e_i}T, Je_j\right)\\
&= \sum_{i,j=1}^{4}\left(D^g_{e_i}T, e_j\right)\left(D^g_{Je_i}T, Je_j\right)\\
&= \frac{1}{2}\,(d\theta, J(d\theta)),
\end{aligned}
$$

again since T is Killing, so that $D^g\,\theta = \frac{1}{2}\,d\theta$, and by using (8).

Finally,

$$\begin{aligned}\frac{1}{2}\,(d\theta, J(d\theta)) &= -\frac{1}{2}\,(d\theta, *\,d\theta)\\ &= -\,\frac{1}{2}\,*\,(d\theta\wedge d\theta) = -\frac{1}{2}\,*\,d(\theta\wedge d\theta) = \frac{1}{2}\,*\,d*(*(\theta\wedge d\theta))\\ &= -\,\frac{1}{2}\,\delta(*(\theta\wedge d\theta)) = -\frac{1}{2}\delta(T\lrcorner *\,d\theta).\end{aligned}$$

Bringing together this result with (27), we eventually get (21). □

Proof of Theorems 2-3. By (15) of (F3) and (16) of Lemma 1 (where $\delta\theta$ is zero since the metric g is standard), we get:

$$(R^g(\Omega))(T, JT) = \frac{\kappa}{4}\,|T|^2, \tag{30}$$

where κ is constant by (F2).

¿From (F1) and Lemma 2, we thus get :

$$\kappa|T|^2 = 0. \tag{31}$$

Since κ is constant, we obtain the following alternative :

(i) $\theta = 0$ (everywhere), or

(ii) $\kappa = 0$.

In the former case, D coincides with D^g by (1), hence (g, J) is Kähler since $D^gJ = DJ = 0$, in fact Einstein -Kähler by (15).

In the latter case, (15) reduces to:

$$\mathrm{Ric}^g = \frac{1}{2}\,\left(|\theta|^2 g - \theta\otimes\theta\right). \tag{32}$$

In particular, $\mathrm{Ric}^g(T) = 0$. Since T is Killing, it follows that T is in fact g-parallel. We then infer from (32) that g is locally isometric, up to a constant, to the product $S^1_a \times S^3$, where S^1_a is the circle of radius a, for some $a > 0$, and S^3 the 3-sphere of sectional cuvature 1. Then, we infer from (1) that D is flat, so that the complex surface (M, J) is a Hopf surface (cf. [10] or [4]).

References

1. A.L. BESSE, *Einstein manifolds.* Ergebnisse der Mathematik **10** (1987), Springer-Verlag.
2. S.K. DONALDSON-P.B. KRONHEIMER, *The geometry of four-manifolds*, Clarendon Press, Oxford, (1990).
3. P. GAUDUCHON, *La topologie d'une surface hermitienne d'Einstein* , C.R. Acad. Sci. Paris, **290** (17 mars 1980), A 509-512.
4. P. GAUDUCHON, *Structures de Weyl-Einstein, espaces de twisteurs et variétés de type* $S^1 \times S^3$, J. reine ang. Math. **469** (1995), 1-50.
5. G. GANCHEV-S.IVANOV, *Compact Hermitian surfaces of Einstein type with respect to the Hermitian connection*, Monatshefte Math. **123** (1997), 53-59.
6. N.J. HITCHIN, Review of [Gal] in Math. Reviews 1981, 81e/53052.
7. S. KOBAYASHI, *Differential geometry of complex vector bundles*, Princeton University Press, 1987.

8. H. PEDERSEN, Y.S. POON and A. SWANN, *The Einstein-Weyl equations in Complex and Quaternionic Geometry*, J. Diff. Geom. Appl.3, N^o 4 (1993), 309-321.
9. K.P. TOD, *Compact 3-dimensional Einstein-Weyl structures*, J. London Math. Soc. (**2**)45 (1992), 341-351.
10. I. VAISMAN, *Some curvature properties of complex surfaces*, Ann. Mat. Pura Appl. **32** (1982), 1-18.

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