ON THE FUNDAMENTAL THEOREM FOR NON-DEGENERATE COMPLEX AFFINE HYPERSURFACE IMMERSIONS

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Abstract

Two geometric versions of the fundamental theorem for non-degenerate complex affine hypersurface immersions are proved. We consider non-degenerate complex affine hypersurface immersions with complex transversal connection form (or equivalently, with holomorphic normalization) and prove that the conormal map is a holomorphic map. This considerations inspired the definitions of complex semi-compatible and complex semi-conjugate connections. This allows us to formulate the integrability conditions of the fundamental theorem, on one hand in terms of the induced connection, which has to be complex semi-compatible and dual H-projective flat, and on the other hand, in terms of its semi-conjugate connection, which has to be H-projective flat. Using this results, we formulate the conditions of the fundamental theorem in terms of any H-projective flat complex affine connection.

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1 Introduction

In the real affine differential geometry, the famous theorem of J.Radon gives a necessary and sufficient conditions for an affine 2-manifold to be affinely immersible in $\mathbf{R}^3$. This fundamental theorem received a new geometric formulation for all dimensions in [3] (see also [20, 11, 8]). A different formulation of this fundamental problem is given by Norden (see [18], Appendix by Norden).

In the complex case, for a given complex hypersurface $\mathbf{M}^{2n}$ (of real dimension $2n$) in complex affine space $\mathbf{C}^{n+1}$ and a given transversal vector field there exists a complex affine connection ∇, called the induced connection and a symmetric totally pure (0,2) tensor h, called the second fundamental form and a 1-form τ, called the transversal connection form.

A fundamental question is, conversely, given a complex affine connection ∇ and a symmetric totally pure tensor h on a complex manifold $\mathbf{M}^{2n}$ can $\mathbf{M}^{2n}$ be locally holomorphically immersed in $\mathbf{C}^{n+1}$ in such a way that (∇, h) is realized as the induced structures?

In general, this problem was recently solved in [15].

The aim of the present paper is to formulate and prove two geometric versions of the fundamental theorem for non-degenerate holomorphic hypersurface immersions.

We consider non-degenerate complex affine hypersurface immersions in $\mathbf{C}^{n+1}$ with complex transversal connection form (or equivalently, with holomorphic normalization) and prove that the conormal map is a holomorphic immersion in the dual complex space $\mathbf{C}_{n+1} - \{0\}$ with holomorphic centroaffine normalization. Thus we determine another complex connection ∇^* as the induced connection by the holomorphic conormal immersion.

We formulate in Theorem 5.1 the conditions of the fundamental theorem for non-degenerate complex hypersurface immersion in terms of the connection ∇^*, which has to be holomorphically projective flat and in terms of the connection ∇, which has to be dual H-projective flat.

We define two real-valued support functions using the properties of the conormal immersion and we determine the second fundamental form by the connection ∇^* and the support functions. We formulate in Theorem 6.5 the conditions of the fundamental theorem for non-degenerate complex affine immersion in terms of any holomorphically projectively flat complex affine connection.

2 Affine immersions

In this section, we recall several definitions and preliminary facts on affine hypersurface theory. For more details, see [1, 2, 5, 4, 14, 13].

Throughout the paper all the objects are assumed to be of class C^∞ in the real sense, unless otherwise stated.

Let $\mathbf{M}^{2n}$ be a complex manifold of real dimension $2n$ $(2n \geq 4)$ and let $f : \mathbf{M}^{2n} \to \mathbf{C}^{n+1}$ be a holomorphic immersion of $\mathbf{M}^{2n}$ into the standard complex affine space $\mathbf{C}^{n+1}$ equipped with the usual flat complex affine connection D. We shall denote by J the complex structure on $\mathbf{M}$ as well as the standard one on $\mathbf{C}^{n+1} \cong (\mathbf{R}^{2n+2}, J)$.

Let ξ be an arbitrary local field of transversal vectors to $f(\mathbf{M}^{2n})$. For any vector fields X and Y on $\mathbf{M}^{2n}$, we write

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi + k(X,Y)J\xi, \tag{2.1}$$

thus defining an affine torsion-free connection ∇ and symmetric tensors h and k of type (0,2). Since f is holomorphic and J is D-parallel, we get that ∇ is complex affine connection (i.e. $\nabla J = 0$) and

$$h(JX, Y) = h(X, JY) = -k(X, Y) \tag{2.2}$$

The tensor h is called *second fundamental form* of (f, ξ). The conditions (2.2) imply that the second fundamental form is a totally pure tensor.

If $(\mathbf{M}^{2n}, J, \nabla)$ is a complex manifold with a torsion-free complex affine connection ∇, we call a holomorphic immersion f a *complex affine immersion* if there exists locally a transversal vector field ξ such that (2.1) holds.

We can also write

$$D_X\xi = -f_*(SX) + \tau(X)\xi + \kappa(X)J\xi, \tag{2.3}$$

where S is the affine shape operator, τ and κ are the 1-forms defining the transversal connection for the immersion (f, ξ). The transversal connection is said to be with complex (resp. anti-complex) transversal connection form if

$$\kappa(X) = -\tau(JX), \qquad (\text{resp.} \kappa(X) = \tau(JX)) \tag{2.4}$$

for every X tangent to $\mathbf{M}$.

A transversal vector field ξ is holomorphic (resp. anti-holomorphic) iff $SJ = JS$ and the first equality of (2.4) (resp. $SJ = -JS$ and the second equality of (2.4)) holds. If ξ is anti-holomorphic the immersion (f, ξ) is called affine Kaehler immersion [13, 14].If $\tau = 0$ then ξ is called equiaffine and if ξ is holomorphic and $\tau = \kappa = 0$, then ξ is called holomorphic equiaffine.

If we change ξ to $\bar{\xi} = \phi\xi + \psi J\xi + Q$, where Q is a smooth tangent vector field and ϕ, ψ are smooth functions, nowhere-vanishing simultaneously on $\mathbf{M}^{2n}$, then the corresponding objects change as follows:

$$\phi\bar{h} - \psi\bar{k} = h; \qquad \psi\bar{h} + \phi\bar{k} = k; \tag{2.5}$$

$$\nabla_X Y = \bar{\nabla}_X Y + \bar{h}(X, Y)Q + \bar{k}(X, Y)JQ; \tag{2.6}$$

$$\phi\bar{\tau}(X) - \psi\bar{\kappa}(X) = X\phi + \phi\tau(X) - \psi\kappa(X) + h(X, Q); \tag{2.7}$$

$$\psi\bar{\tau}(X) + \phi\bar{\kappa}(X) = X\psi + \psi\tau(X) + \phi\kappa(X) + k(X, Q); \tag{2.8}$$

Let R denote the curvature tensor of ∇. We have the following fundamental equations:

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY - h(Y, JZ)JSX + h(X, JZ)JSY. \tag{2.9}$$

$$\begin{aligned}(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = -\tau(X)h(Y, Z) + \tau(Y)h(X, Z) - \\ \kappa(X)h(Y, JZ) + \kappa(Y)h(X, JZ).\end{aligned} \tag{2.10}$$

$$(\nabla_X S)Y - (\nabla_Y S)X = \tau(X)SY - \tau(Y)SX + \kappa(X)JSY - \kappa(Y)JSX \tag{2.11}$$

$$h(X, SY) - h(Y, SX) = d\tau(X, Y). \tag{2.12}$$

$$h(JX, SY) - h(JY, SX) = d\kappa(X, Y). \tag{2.13}$$

The equation (2.9) is *the Gauss equation*, (2.10) is *the Codazzi I equation*, (2.11) is *the Codazzi II equation*, (2.12) is *the Ricci I equation* and (2.13) is *the Ricci II equation*.

Conversely, these five equations are the sufficient integrability conditions for the existence of a local complex affine immersion of $\mathbf{M}^{2n}$. This result was proved by B.Opozda in [15, 16]. The existence theorem for affine Kaehler immersions is proved in [14]. We shall formulate the existence theorem:

Theorem 2.1 *[15, 16] Let $(\mathbf{M}^{2n}, J, \nabla)$ be a complex $2n$-dimensional manifold with a complex structure J endowed with a torsion-free complex affine connection ∇, a symmetric bilinear form h with the property (2.2), (1,1) -tensor S and 1-forms τ and κ satisfying equations (2.9) $\div$ (2.13). Then there exists a complex affine immersion (f, ξ) of $(\mathbf{M}^{2n}, J, \nabla)$ into $\mathbf{C}^{n+1}$ such that h, S, τ, κ are the induced objects of (f, ξ). Such a complex affine immersion is unique up to a holomorphic affine transformation on $\mathbf{C}^{n+1}$.*

If $\mathbf{M}^{2n}$ is assumed to be simply connected, all the results above hold in a global setting.

If $\kappa = -\tau \circ J$ in the conditions of Theorem 2.1 we have the existence theorem for complex affine immersions with complex transversal connection form.

A complex affine immersion (f, ξ) is said to be *non-degenerate* if the induced second fundamental form h is non-degenerate. It is easy to see from (2.5) that this condition is independent of the choice of ξ.

A non-degenerate symmetric $(0,2)$ tensor h on a complex manifold $(\mathbf{M}^{2n}, J)$ is said to be *a complex Riemannian metric* if h and the complex structure J are compatible in the following way: $h(JX, Y) = h(X, JY)$. This condition implies that h has to be of signature (n, n). The triple $(\mathbf{M}^{2n}, J, h)$ is called *complex Riemannian manifold.* By extending h as a complex bilinear function on complex tangent vector fields we see that h is completely determined by its values on $(1,0)$-vector fields, namely for $V = X - iJX$, $W = Y - iJY$ we have $h(V, W) = 2(h(X, Y) - ih(X, JY)), h(V, \bar{W}) = 0$, where $\bar{W} = Y + iJY$ denotes complex conjugate. If we consider the complex valued function $f = \phi + i\psi$ then the transformations (2.5) can be written in the following way $h = f\bar{h}$. Transformations of complex Riemannian metric of the form (2.5) are called *complex conformal transformations.*

A complex Riemannian metric h is said to be *complex analytic Riemannian metric* if the function $h(V, W)$ is a holomorphic function whenever V, W are holomorphic $(1,0)$-vector fields. Complex analytic Riemannian metrics can be characterized by the following fact:

A complex Riemannian metric h is a complex analytic Riemannian metric iff the Levi-Chivita connection ∇^h of h is a complex connection.

This fact follows from considerations in [6]. We give a proof for completeness. We extend ∇^h by a complex linearity on complex tangent vector fields. The connection ∇^h is a complex connection iff $h(\nabla^h_{V_1} \bar{V}_2, V_3) = h(\nabla^h_{\bar{V}_1} V_2, \bar{V}_3) = 0$ for every $(1,0)$-vector fields V_1, V_2, V_3. If V_1, V_2, V_3 are mutually commuting complex vector fields then $2h(\nabla^h_{V_1} V_2, V_3) = V_1 h(V_2, V_3) + V_2 h(V_1, V_3) - V_3 h(V_1, V_2)$. Since $\mathbf{M}$ is a complex manifold the last two equalities imply that ∇^h is a complex connection iff h is a complex analytic Riemannian metric.

Complex analytic Riemannian manifolds have been studied in [17, 12].

3 Holomorphic normalization

The second fundamental form of every non-degenerate complex affine immersion is a complex Riemannian metric by (2.2). We study non-degenerate complex affine immersions with complex analytic second fundamental form.

Definition A non-degenerate complex affine immersion (f, ξ) is said to be *with holomorphic normalization* if the induced second fundamental form h is a complex analytic Riemannian metric.

Proposition 3.1 *Let (f, ξ) be a non-degenerate complex affine immersion. Then the following conditions are equivalent:*

a) (f, ξ) is with holomorphic normalization;

b) (f, ξ) is with complex transversal connection form.

Proof: By the well known theorem (see [19], p.132) for the Levi-Civita connection ∇^h of the complex Riemannian metric h we have:

$$2h(\nabla^h_X Y, Z) = 2h(\nabla_X Y, Z) + (\nabla_X h)(Y, Z) + (\nabla_Y h)(X, Z) - (\nabla_Z h)(Y, X).$$

Using the Codazzi I equation (2.10) we get

$$2h(\nabla^h_X Y, Z) = 2h(\nabla_X Y, Z)+ \tag{3.14}$$

$$(\nabla_X h)(Y, Z) - \tau(Y)h(X, Z) + \tau(Z)h(X, Y) - \kappa(Y)h(X, JZ) + \kappa(Z)h(X, JY).$$

Now, (3.14) implies that ∇^h is a complex connection iff $\kappa = -\tau \circ J$ **Q.E.D.**

The following result follows easily from Proposition 3.1 and formulae (2.7), (2.8).

Proposition 3.2 *Let (f, ξ) be a non-degenerate complex affine immersion with holomorphic normalization and let $\bar{\xi} = \phi\xi + \psi J\xi + Q$ be another transversal vector field, where Q is a tangent vector field and ϕ, ψ are real functions, nowhere vanishing simultaneously. Then $(f, \bar{\xi})$ is with holomorphic normalization iff the functions ϕ, ψ satisfy the following condition:*

$$d\psi = -d\phi \circ J \tag{3.15}$$

Complex conformal transformations satisfying the condition (3.15) are called *complex analytic conformal (analytic conformal) transformations* (see [12, 6]).

Further we need the notions of the conormal map of a given non-degenerate complex affine immersion (f, ξ) with holomorphic normalization. Let $(\mathbf{R}_{2n+2}, J) \cong \mathbf{C}_{n+1}$ denote the dual complex vector space to the complex vector space $(R^{2n+2}, J) \cong \mathbf{C}^{n+1}$ associated to the complex affine space $\mathbf{C}^{n+1}$with the standard complex structure. For each $p \in \mathbf{M}^{2n}$, let v_p be a covector ($v_p \in \mathbf{R}_{2n+2}$), uniquely determined by the conditions:

$$v_p(\xi_p) = 1, \qquad v_p(J\xi_p) = 0 \qquad v_p(f_* Y) = 0, \tag{3.16}$$

for every vector Y at $p \in \mathbf{M}^{2n}$.

The map $v : p \in \mathbf{M}^{2n} \to v_p \in \mathbf{R}_{2n+2}$ is called the conormal map of (f, ξ), and v is called the affine conormal. The conormal map of a non-degenerate complex affine immersion with holomorphic equiaffine transversal vector field is determined in [4].

Proposition 3.3 *Let (f, ξ) be a non-degenerate complex affine immersion with holomorphic normalization. Then the induced conormal map v is a holomorphic map.*

Proof: Using (2.1) and (2.3) from (3.16) we obtain:

$$v_*(Y)(\xi) = -\tau(Y); \qquad v_*(Y)(J\xi) = \kappa(Y); \qquad v_*(JY)(\xi) = -\tau(JY). \tag{3.17}$$

$$v_*(Y)(f_*X) = -h(Y, X); \qquad v_*(JY)(f_*X) = v_*(Y)(f_*JX) = -h(Y, JX). \tag{3.18}$$

Since (f, ξ) is with holomorphic normalization Proposition 3.1, (3.17) and (3.18) imply $J \circ v_* = v_* \circ J$ and the conormal map is a holomorphic map. **Q.E.D.**

From (3.17) and (3.18) it follows that the conormal map is a holomorphic immersion $\mathbf{M}^{2n} \to \mathbf{R}_{2n+2} - \{0\}$, as f is non-degenerate. Then (3.16), (3.17) and (3.18) show that v itself is a transversal field to $v(\mathbf{M}^{2n})$. We consider $v(\mathbf{M}^{2n})$ as a complex centro-affine hypersurface (by taking $-v$ as a transversal vector at each point). We can then write the structure equation :

$$D_X(v_*(Y)) = v_*(\nabla^*_X Y) - h^*(X,Y)v - k^*(X,Y)Jv, \tag{3.19}$$

where ∇^* is another torsion free complex affine connection on $\mathbf{M}^{2n}$ and h^*, k^* are other symmetric (0,2) tensors on $\mathbf{M}^{2n}$ satisfying the condition (2.2). We have

Proposition 3.4 *Let (f, ξ) be a non-degenerate complex affine immersion with complex transversal connection form τ. Then the following formulae are valid:*

$$h^*(X,Y) - h(X,SY) = (\nabla^*_Y \tau)X - \tau(Y)\tau(X) + \tau(JY)\tau(JX); \tag{3.20}$$

$$Xh(Y,Z) = h(\nabla_X Y, Z) + h(Y, \nabla^*_X Z) + \tau(Z)h(X,Y) - \tau(JZ)h(JX,Y). \tag{3.21}$$

Proof: Differentiating in (3.16) and (3.18), applying (2.1), (2.3), (3.19), (3.17) and (3.18) we get the proof by straightforward calculations. **Q.E.D.**

Corollary 3.5 *Let (f, ξ) be a non-degenerate complex affine immersion with $\tau = \kappa = 0$. Then ξ is a holomorphic equiaffine transversal vector field.*

Proof: From (3.20) we have $h^*(X,Y) = h(X,SY)$. Using the condition (2.2) we get $SJ = JS$. **Q.E.D.**

According to [5] for every non-degenerate complex affine hypersurface immersion there always exists a holomorphic equiaffine transversal vector field. If the transversal vector field is holomorphic equiaffine then the induced second fundamental form is a complex analytic Riemannian metric. Thus, every non-degenerate complex affine hypersurface immersion admits a holomorphic normalization. For example, let G be the hypersurface of $\mathbf{C}^{n+1}$ described by $z_{n+1} = \frac{1}{2}(z_1^2 + ... + z_n^2)$. The vectors $\frac{\partial}{\partial z_\alpha} = (0, ..., 0, 1, 0, ..., 0, z_\alpha)$, where 1 occurs at the α'th place, give a basis for the $(1,0)$-space of G and $\xi = (0, ...0, 1)$ is a holomorphic equiaffine transversal vector field. The corresponding second fundamental form $h(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}) = \delta_{\alpha\beta}$, where $\delta_{\alpha\beta}$ are the Kronrcker's symbols, is a complex analytic Riemannian metric with a flat Levi-Civita connection.

Remark 3.6 *It is clear by Proposition 3.2 that for a non-degenerate complex affine immersion with holomorphic normalization the transversal vector field may not be a holomorphic vector field.*

4 Projective flatness.

In this section we give a short review on H-projective structures and dual H-projective structures. We refer for the proofs to [21, 22] and [10], respectively.

In this section, let $(\mathbf{M}, J, h)$ be a 2n-dimensional complex-Riemannian manifold. Let ∇ be a torsion-free complex affine connection and let $c : I_\epsilon \to \mathbf{M}, I_\epsilon = (-\epsilon, \epsilon), \epsilon > 0$, be a smooth curve on $\mathbf{M}$ with tangent vector field $\dot{c}$. Let $\omega_{\dot{c}}$ be the 1-form corresponding to the tangent vector $\dot{c}$ (with respect to the metric h), defined by $\omega_{\dot{c}}(X) := h(\dot{c}, X)$. The curve c

is said to be *H-planar curve* if the space span$\{\dot{c}, J\dot{c}\}$ is parallel along the curve; the curve c is said to be *dual H-planar curve* if the space span$\{\omega_{\dot{c}}, \omega_{J\dot{c}}\}$ is parallel along the curve.

Two torsion-free complex affine connections are said to be *H-projectively* (resp. *dual H-projectively) equivalent* if their H-planar (resp. dual H-planar) curves coincide.

Two torsion-free affine connections ∇ and ∇' on $\mathbf{M}$ are H-projectively equivalent, iff $\nabla'_X Y = \nabla_X Y + \alpha(X)Y + \alpha(Y)X - \alpha(JY)JX - \alpha(JX)JY$, where α is a 1-form on $\mathbf{M}$.

Two torsion-free affine connections ∇ and ∇' on $\mathbf{M}$ are dual H-projectively equivalent, iff $\nabla'_X Y = \nabla_X Y - h(X,Y)Q + h(X,JY)JQ$, where Q is a vector field on $\mathbf{M}$.

Given a torsion-free complex affine connection ∇ on $\mathbf{M}$ let Ric denotes its Ricci tensor. The H-projective curvature tensor HW of ∇ is defined by

$$HW(X,Y)Z = R(X,Y)Z - N(X,Z)Y + N(Y,Z)X - (N(X,Y) - N(Y,X))Z+$$

$$+N(X,JZ)JY - N(Y,JZ)JX + (N(X,JY) - N(Y,JX)JZ,$$

where

$$N(X,Y) = -\frac{1}{2n+2}\{Ric(X,Y) + \frac{1}{2n-2}[Ric(X,Y) + Ric(Y,X)-$$
$$-Ric(JX,JY) - Ric(JY,JX)]\}.$$

If the Ricci tensor is symmetric and pure the last equality implies that the tensor N is equal to the normalized Ricci tensor γ defined by $2(n-1)\gamma = Ric$.

We also need to consider the Codazzi equation for N:

$$(\nabla_X N)(Y,Z) = (\nabla_Y N)(X,Z). \tag{4.22}$$

An affine connection ∇ is said to be *H-projectively* (resp. *dual H-projectively) flat* if around each point there is a H-projective (resp. dual H-projective) change of ∇ to a flat affine connection. The following result is well known (Y.Tashiro), (see [21, 22]),

Theorem 4.1 *[21] Let $(\mathbf{M}, J)$ be a complex manifold with a torsion-free complex affine connection ∇. Then:*

If $\dim \mathbf{M} \geq 6$, then ∇ is H-projectively flat iff $HW = 0$. In this case (4.22) also holds;

If $\dim \mathbf{M} = 4$, then ∇ is H- projectively flat iff $HW = 0$ and (4.22) holds.

For $\dim \mathbf{M} = 4$ we also have

Corollary 4.2 *Let $(\mathbf{M}, J)$ be a 4-dimensional complex manifold with a torsion-free complex affine connection ∇. Suppose that the curvature tensor R of ∇ is totally pure i.e. $R(JX,Y)Z = R(X,JY)Z = R(X,Y)JZ = JR(X,Y)Z$. Then the connection ∇ is H-projectively flat iff the condition (4.22) holds.*

Proof: Since $dim\mathbf{M} = 4$ and the curvature tensor R is totally pure then the H-projective tensor HW of ∇ is identically zero. Then Theorem 4.1 implies the assertion. **Q.E.D.**

H-projective transformations on complex Riemannian manifolds are considered also in [7].

A torsion-free complex affine connection ∇ on $(\mathbf{M}, J, h)$ is said to be *complex semi-compatible with the metric h* by a 1-form τ if the following identity holds

$$(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = -\tau(X)h(Y,Z) + \tau(Y)h(X,Z)+ \tag{4.23}$$

$$\tau(JX)h(Y,JZ)-\tau(JY)h(X,JZ).$$

If the 1-form $\tau=0$ then ∇ is said to be *compatible* with the metric h.

In [10] it is proved that if a complex Riemannian manifold admits a torsion-free complex affine connection which is complex semi-compatible with the metric then the metric has to be complex analytic Riemannian manifold.

An affine connection ∇^* on a complex Riemannian manifold $(\mathbf{M},J,h)$ is said to be *complex semi-conjugate to* ∇ by a 1-form τ if the equality (3.21) holds i.e.

$$Xh(Y,Z)=h(\nabla_X Y,Z)+h(Y,\nabla^*_X Z)+\tau(Z)h(X,Y)-\tau(JZ)h(JX,Y).$$

If the 1-form $\tau=0$ then the connections ∇^* and ∇ are said to be *conjugate connections.*

A complex semi-conjugate connection ∇^* is always a complex connection and the connection ∇^* is torsion-free iff the connection ∇ is complex semi-compatible with h. If ∇ is an affine connection complex semi-compatible with the metric h by τ then there exists a unique affine connection ∇^* complex semi-conjugate to ∇ by τ; this unique connection ∇^* is always complex compatible with h. The difference (1.2) tensor between ∇ and ∇^*, say $K=\nabla^*-\nabla$, is determined by the equality [10].

$$h(K(X,Y),Z)=(\nabla_X h)(Y,Z)-\tau(Y)h(X,Z)+\tau(JY)h(JX,Z). \tag{4.24}$$

The connection between the curvature tensors R of ∇ and R^* of ∇^* is given by the following formula [10]:

$$\begin{aligned} &h(R^*(X,Y)Z,U)+h(R(X,Y)Z,U)= \\ &h(X,U)\left[(\nabla^*_Y\tau)Z-\tau(Y)\tau(Z)+\tau(JY)\tau(JZ)\right]- \\ &h(Y,U)\left[(\nabla^*_X\tau)Z-\tau(X)\tau(Z)+\tau(JX)\tau(JZ)\right]- \\ &h(X,JU)\left[(\nabla^*_Y\tau)JZ-\tau(JY)\tau(Z)-\tau(Y)\tau(JZ)\right]+ \\ &h(Y,JU)\left[(\nabla^*_X\tau)JZ-\tau(JX)\tau(Z)-\tau(X)\tau(JZ)\right]. \end{aligned} \tag{4.25}$$

Let $(\mathbf{M},h,J)$ be a $2n$-dimensional complex analytic Riemannian manifold and ∇ be an affine connection complex semi-compatible with h. The dual H-projective curvature tensor DHW of ∇ is defined by [10]:

$$\begin{aligned} DHW(X,Y)Z&=R(X,Y)Z-h(Y,Z)M(X)+h(X,Z)M(Y)+ \\ &h(Y,JZ)JM(X)-h(X,JZ)JM(Y), \end{aligned} \tag{4.26}$$

The tensor M is given by:

$$M(X)=-Ric^{\#}(X)+\frac{tr_h(Ric)}{2(n-1)}X-\frac{tr_h(J\circ Ric)}{2(n-1)}, \tag{4.27}$$

where $\mathrm{Ric}^{\#}(X)$ is the Ricci operator, corresponding to the Ricci tensor Ric, defined by $h(Ric^{\#}(X),Y):=Ric(X,Y)$ and tr_h denote the trace with respect to h.

We have to consider the Codazzi II equation for M:

$$\begin{aligned} (\nabla_X M)Y-(\nabla_Y M)X&=\tau(X)M(Y)-\tau(Y)M(X)- \\ &\tau(JX)JM(Y)+\tau(JY)JM(X). \end{aligned} \tag{4.28}$$

We need the following results from [10]

Theorem 4.3 *[10] The dual H-projective tensor of an affine connection complex semi-compatible with a complex analytic Riemannian metric is an invariant of the class of dual H-projective transformations.*

Theorem 4.4 *[10] Let $(\mathbf{M}, J, h)$ be a complex analytic Riemannian manifold and ∇ be an affine connection complex semi compatible with h by τ. Then:*

i) If $\dim \mathbf{M} \geq 6$, then ∇ is dual H-projectively flat iff $DHW = 0$. In this case (4.28) holds and $trR = -2d\tau, \quad tr(J \circ R) = -2d(\tau \circ J)$.

ii) For $\dim \mathbf{M} = 4$, let the curvature tensor R of ∇ satisfies $trR = -2d\tau, \quad tr(J \circ R) = -2d(\tau \circ J)$. Then ∇ is dual H- projectively flat iff $DHW = 0$ and (4.28) holds.

5 Fundamental theorem.First version.

In this section we formulate and prove one of our main results.

Theorem 5.1 *Let $(\mathbf{M}, J, h, \nabla)$ be a $2n$-dimensional connected and simply connected complex analytic Riemannian manifold with a complex structure J, a complex analytic metric h and a torsion-free complex affine connection ∇. Let R be the curvature tensor of ∇ and τ be a 1-form on $\mathbf{M}$. Then the following conditions are equivalent:*

i) There exists a non-degenerate complex affine immersion $f : \mathbf{M}^{2n} \to \mathbf{C}^{n+1}$ with complex transversal connection form τ such that the induced fundamental form and the induced connection of the immersion are h and ∇, respectively. Such an immersion is unique up to a holomorphic affine transformation on C^{n+1}.

ii) The affine connection ∇ is complex semi-compatible with h by τ and the unique affine connection ∇^* complex semi-conjugate to ∇ by τ is H-projectively flat and has symmetric and pure Ricci tensor.

iii) The connection ∇ is complex semi-compatible with h by τ.

For $2n \geq 6$ the connection ∇ is dual H-projective flat affine connection.

For $2n = 4, \nabla$ satisfies $trR = -2d\tau, \quad tr(J \circ R) = -2d(\tau \circ J)$ and ∇ is a dual H-projective flat affine connection.

Proof: First we prove the equivalence of i) and ii).

Let the condition i) of Theorem 5.1 holds. Then $\kappa = -\tau \circ J$. From the Codazzi I equation (2.10) we have that ∇ is complex semi-compatible with h by τ. Let ∇^* denote the affine connection complex semi-conjugate to ∇ by τ. The connection ∇^* is the connection induced by the centroaffine holomorphic conormal map $(v, -v)$. Then the Gauss equation and the Codazzi I equation can be written as follows:

$$R^*(X,Y)Z = -h^*(X,Z)Y + h^*(Y,Z)X + h^*(X,JZ)JY - h^*(Y,JZ)JX. \tag{5.29}$$

$$(\nabla^*_X h^*)(Y,Z) = (\nabla^*_Y h^*)(X,Z), \tag{5.30}$$

where h^* is the second fundamental form induced by $(v, -v)$. From (5.29) we get by a contraction:

$$Ric(R^*) = 2(n-1)h^*, \tag{5.31}$$

i.e. h^* is equal to the normalized Ricci tensor γ^* of ∇^*. Hence, the Ricci tensor $Ric(R^*)$ of ∇^* is symmetric and pure. Now Theorem 4.1 and Corollary 4.2 imply the connection ∇^* is H-projective flat which implies ii).

Conversely, assume that ∇ is complex semi-compatible with h by τ and the affine connection ∇^*, complex semi-conjugate to ∇ by τ is H-projectively flat with symmetric and pure Ricci tensor. Then the Codazzi I equation (2.10) is valid. By Theorem 4.1 the curvature tensor R^* of ∇^* is given by (5.29) and (5.31), where $h^* = \gamma^*$ is the normalized Ricci tensor of ∇^*. Since h is non-degenerate, we can define an operator S by the equality:

$$h(X, SY) = h^*(X, Y) - (\nabla^*_Y \tau)(X) + \tau(X)\tau(Y) - \tau(JX)\tau(JY), \tag{5.32}$$

From (5.32) we get that the Ricci equations (2.12) and (2.13) are valid. From (4.25) using (5.29), (5.31) and (5.32) we obtain the Gauss equation (2.9). Since ∇^* is H-projectively flat with symmetric and pure Ricci tensor then Theorem 4.1 and Corollary 4.2 imply that the equations (5.30) are valid. Differentiating in (5.32) we obtain:

$$\begin{gathered}(\nabla^*_X \nabla^*_Y \tau) Z = (\nabla^*_X h^*)(Y, Z) - (\nabla^*_X \bar{h})(Y, Z) + \\ \tau(Z)\left[h^*(X, Y) - \bar{h}(X, Y)\right] + \tau(Y)\left[h^*(X, Z) - \bar{h}(X, Z)\right] - \\ \tau(JZ)\left[h^*(X, JY) - \bar{h}(X, JY)\right] - \tau(JY)\left[h^*(X, JZ) - \bar{h}(X, JZ)\right] + \\ 2\tau(X)\tau(Y)\tau(Z) - 2\tau(X)\tau(JY)\tau(JZ) - 2\tau(JX)\tau(Y)\tau(JZ) - 2\tau(JX)\tau(JY)\tau(Z)\end{gathered} \tag{5.33}$$

where we used the following substitution:

$$\bar{h}(X, Y) = h(SX, Y). \tag{5.34}$$

From the Ricci identities using (5.29),(5.33) and (5.30) we get:

$$\begin{gathered}(\nabla^*_X \bar{h})(Y, Z) - (\nabla^*_Y \bar{h})(X, Z) = -\tau(Y)\bar{h}(X, Z) + \tau(X)\bar{h}(YZ) + \\ \tau(JY)\bar{h}(X, JZ) - \tau(JX)\bar{h}(YJZ) - \\ \tau(Z)\left[\bar{h}(X, Y) - \bar{h}(Y, X)\right] + \tau(JZ)\left[\bar{h}(X, JY) - \bar{h}(Y, JX)\right].\end{gathered} \tag{5.35}$$

From (5.34) using (3.21) we obtain:

$$\begin{gathered}(\nabla^*_X \bar{h})(Y, Z) = X\bar{h}(Y, Z) - \bar{h}(\nabla^*_X Y, Z) - \bar{h}(Y, \nabla^*_X Z) = \\ h(\nabla_X SY, Z) - h(S\nabla^*_X Y, Z) + \tau(Z)h(SY, X) - \tau(JZ)h(SY, JX).\end{gathered}$$

Since ∇ and ∇^* are torsion-free we obtain:

$$\begin{gathered}(\nabla^*_X \bar{h})(Y, Z) - (\nabla^*_Y \bar{h})(X, Z) = h(\nabla_X SY, Z) - h(S\nabla^*_X Y, Z) - \\ h(\nabla_Y SX, Z) + h(S\nabla^*_Y X, Z) + \tau(Z)\left[h(SY, X) - h(SX, Y)\right] - \\ \tau(JZ)\left[h(SY, JX) - h(SX, JY)\right] = h((\nabla_X S)Y, Z) - h((\nabla_Y S)X, Z) + \\ \tau(Z)\left[h(SY, X) - h(SX, Y)\right] - \tau(JZ)\left[h(SY, JX) - h(SX, JY)\right].\end{gathered} \tag{5.36}$$

Substituting (5.36) into (5.35) and using the notations (5.34) we obtain the Codazzi II equation (2.11). Thus, all the conditions of the existence Theorem 2.1 are fulfilled and by the same theorem we get the proof of i).

Now we prove the equivalence of i) and iii).

Let i) hold. From the Gauss equation (2.9), the Codazzi I equation (2.10), the Codazzi II equation (2.11) we have that the conditions of Theorem 4.4 are satisfied. Then from the same theorem ∇ is dual H-projective flat.

For the inverse, let iii) hold. We define an operator $S = M$,where M is given by (4.27). Then Theorem 4.4 implies that all the conditions of the Theorem 2.1 are satisfied which implies i). This completes the proof of the theorem. **Q.E.D.**

Corollary 5.2 *Let* $(\mathbf{M}, J, h, \nabla)$ *be a* $2n$*-dimensional connected and simply connected complex analytic Riemannian manifold with a complex structure* J*, a complex analytic metric* h *and a torsion-free complex affine connection* ∇*. Then the following conditions are equivalent:*

i) There exists a non-degenerate complex affine immersion $f : \mathbf{M}^{2n} \to \mathbf{C}^{n+1}$ *with holomorphic equiaffine transversal vector field* ξ *such that the induced second fundamental form and the induced connection of the immersion* (f, ξ) *are* h *and* ∇*, respectively. Such an immersion is unique up to a holomorphic affine transformation on* C^{n+1}*.*

ii) The affine connection ∇ *is compatible with* h *and the unique affine connection* ∇^* *conjugate to* ∇ *is H-projectively flat and has symmetric and pure Ricci tensor.*

iii) The connection ∇ *is compatible with* h *.*

For $2n \geq 6$ *the connection* ∇ *is dual H-projective flat affine connection.*

For $2n = 4$*, the connection* ∇ *has symmetric and pure Ricci tensor and* ∇ *is dual H-projective flat affine connection.*

Proof: The equivalencies of i) and ii) follows immediately from Theorem 5.1. The equivalencies of i) and iii) follows from Theorem 5.1 and Corollary 3.5. **Q.E.D.**

Remark 5.3 *Let* $\{h\}$ *denote the conformal class of complex Riemannian metrics i.e. the metrics, obtained by* h *with complex conformal transformations. Let* ∇ *be an affine connection complex semi-compatible with some metric of the class* $\{h\}$ *and* $\bar{\nabla}$ *be an affine connection dual H-projective equivalent to* ∇*. From (2.5) and (2.6) it follows that the connection* $\bar{\nabla}$ *is complex semi-compatible with the metrics of* $\{h\}$ *iff the complex conformal transformations (2.5) are complex analytic conformal transformations.*

Now, let $\{h\}$ *denote the complex analytic conformal class of complex Riemannian metrics, i.e. the metrics, obtained by* h *with complex analytic conformal transformations. If one of the metrics of* $\{h\}$ *is complex analytic then Proposition 3.2 implies that every metric of* $\{h\}$ *is complex analytic. From (4.26) and (4.27) it follows that the dual H-projective tensor* DHW *is an invariant of the class of complex analytic conformal transformations. If* $f : \mathbf{M}^{2n} \to \mathbf{C}^{n+1}$ *is a non-degenerate complex affine immersion with holomorphic normalization then Theorem 4.3 implies that the dual H-projective tensor* DHW *does not depend on the holomorphic normalization. In particular, for non-degenerate complex affine immersion with holomorphic normalization, we have* $DHW = 0$*.*

Conversely, (for $2n \geq 6$ *) Theorem 5.1 implies that for a given conformal class* $\{h\}$ *of complex analytic Riemannian metrics and a given class* $\{\nabla\}$ *of affine connections complex semi-compatible with metrics of* $\{h\}$*, there exists a non-degenerate complex affine hypersurface immersion* (f, ξ, τ) *with holomorphic normalization such that* $\{h\}$ *and* $\{\nabla\}$ *are realized as induced structures from the immersion* (f, ξ, τ) *if and only if* $DHW = 0$*.*

Geometric formulations of the fundamental theorem for affine Kaehler immersion is considered in [9].

6 Fundamental theorem. Second version

Let $f : \mathbf{M}^{2n} \to (\mathbf{R}^{2n+2}, J)$ be a non-degenerate complex affine immersion with a transversal vector field ξ and complex transversal connection form τ. Let $v : \mathbf{M}^{2n} \to (\mathbf{R}_{2n+2}, J)$ be its conormal map. Given a point $c \in \mathbf{R}^{2n+2}$ we define the support functions $\rho_c(x)$ and

$\lambda_c(x)$ on $\mathbf{M}$ by setting

$$\rho_c(x) = v(c - f(x)); \qquad \lambda_c(x) = Jv(c - f(x)), x \in \mathbf{M}.$$

Then we can write

$$c - f(x) = \rho(x)\xi_x - \lambda(x)J\xi_x + f_*(Z_X),$$

where Z is a certain tangent vector at x.

For simplicity, take $c = 0$ and write ρ (resp. λ) for this support functions.

Proposition 6.1 *The support functions satisfy the equations:*

$$h(X,Y) = \rho\gamma^*(X,Y) - \lambda\gamma^*(X,JY) + Hess^*_\rho(X,Y); \tag{6.37}$$

$$d\lambda = d\rho \circ J, \tag{6.38}$$

where γ^* is the normalized Ricci tensor of the connection ∇^* induced by the conormal map and $Hess^*$ is the Hessian relative to ∇^*, i.e.

$$2(n-1)\gamma^* = Ric(R^*), \qquad Hess^*_\rho(X,Y) = XY\rho - (\nabla^*_X Y)\rho \tag{6.39}$$

Proof: From $v(f) = -\rho$, we obtain

$$-Y\rho = Y(v(f)) = (D_Y v)f + v(f_*(Y)) = v_*(Y)f. \tag{6.40}$$

Since v is a holomorphic map from (6.40) we get $-JY\rho = v_*(JY)f = -Y\lambda$ which proves (6.38). Using $-f = \rho\xi - \lambda J\xi + f_*(Z)$ and the formulae (3.17), (3.18) we get from (6.40)

$$-Y\rho = \rho\tau(Y) - \lambda\tau(JY) + h(Y,Z).$$

Differentiating (6.40) and using (3.19) we obtain:

$$-XY\rho = -(\nabla^*_X Y)\rho - h(X,Y) + \rho h^*(X,Y) - \lambda h^*(X,JY).$$

Observing that $h^* = \gamma^*$ we obtain (6.37). **Q.E.D.**

The converse is also true

Proposition 6.2 *Let ∇^* be a torsion-free complex, H-projectively flat affine connection with symmetric and pure normalized Ricci tensor γ^*. If ρ and λ are any smooth functions on $\mathbf{M}$, satisfying the condition (6.38) then the (0,2) tensor h of the form (6.37) is a symmetric and pure tensor and satisfies the Codazzi equation*

$$(\nabla^*_X h)(Y,Z) = (\nabla^*_Y h)(X,Z). \tag{6.41}$$

Proof: The conditions (6.37) and (6.38) imply:

$$h(X,JY) = \rho\gamma^*(X,JY) + \lambda\gamma^*(X,Y) + Hess^*_\lambda(X,Y); \tag{6.42}$$

From (6.37) and (6.38) we get that h is symmetric and pure tensor.

By the conditions of the proposition and Theorem 4.1 the curvature tensor R^* of ∇^* is given by (5.29), where $h^* = \gamma^*$ is the normalized Ricci tensor and the condition (5.30) is also valid. The proof of the Codazzi equation follows by straightforward computations using (5.29), (5.30), and (6.39). **Q.E.D.**

Moreover we have

Proposition 6.3 *Let $f : \mathbf{M}^{2n} \to (\mathbf{R}^{2n+2}, J)$ be a non-degenerate complex affine immersion with complex transversal connection form τ and let $v : \mathbf{M}^{2n} \to (\mathbf{R}_{2n+2}, J)$ be the holomorphic conormal immersion with the induced connection ∇^*. Then the solution of the equation (6.37) satisfying the condition (6.38) must be of the form*

$$\rho = v(c - f); \qquad \lambda = Jv(c - f),$$

where c is a constant vector in $\mathbf{R}^{2n+2}$.

Proof: Set $c = \rho\xi - \lambda J\xi - f_*(Z) + f$, where ρ and λ satisfy (6.38) and Z is determined by

(6.43) $$h(Y, Z) = Y\rho + \rho\tau(Y) - \lambda\tau(JY),$$

for all tangent vectors Y.
Using the equations (2.1) $\div$ (2.3), (6.38) and (6.43) for any tangent vector X we get:

$$D_X c = -\rho f_*(SX) + \lambda f_*(JSX) + f_*(X) - f*(\nabla_X Z).$$

Then for any tangent vector Y we have

(6.44) $$h(Y, D_X c) = -\rho h(SX, Y) + \lambda h(SX, JY) + h(X, Y) - h(\nabla_X Z, Y).$$

Using (3.20) from (6.44) we get

(6.45) $$h(Y, D_X c) = -\rho\,[h^*(X, Y) - (\nabla^*_X\tau)Y + \tau(Y)\tau(X) - \tau(JY)\tau(JX)] +$$
$$\lambda\,[h^*(X, JY) - (\nabla^*_X\tau)JY + \tau(JY)\tau(X) + \tau(Y)\tau(JX)] + h(X, Y) - h(\nabla_X Z, Y).$$

From (6.43), (6.38) and the formula $h(\nabla_X Z, Y) = Xh(Y, Z) - h(Z, \nabla_X Y) - (\nabla_X h)(Y, Z)$, we get:

$$h(\nabla_X Z, Y) = XY\rho - (\nabla_X Y)\rho + \rho(\nabla_X\tau)Y - \lambda(\nabla_X\tau)JY + X\rho\tau(Y) -$$
$$X\lambda\tau(JY) - (\nabla_X h)(Y, Z).$$

Applying (4.24), (6.43) and (4.23) we obtain:

$$h(\nabla_X Z, Y) = \rho\,[(\nabla_X\tau)(Y) + \tau(K(X, Y)) - \tau(Y)\tau(X) + \tau(JY)\tau(JX)] + XY\rho -$$
$$[(\nabla_X Y) + K(X, Y)]\,\rho - \lambda\,[(\nabla_X\tau)(JY) + \tau(K(X, JY)) - \tau(JY)\tau(X) - \tau(Y)\tau(JX)]\,.$$

Observing that $\nabla + K = \nabla^*$ we get:

(6.46) $$h(\nabla_X Z, Y) = \rho\,[(\nabla^*_X\tau)(Y) - \tau(Y)\tau(X) + \tau(JY)\tau(JX)] +$$
$$XY\rho - (\nabla^*_X Y)\rho - \lambda\,[(\nabla^*_X\tau)(JY) - \tau(JY)\tau(X) - \tau(Y)\tau(JX)]\,.$$

Substituting (6.46) into (6.45) we have

$$h(Y, D_X c) = -\rho h^*(X, Y) + \lambda h^*(X, JY) - Hess^*_\rho(X, Y) + h(X, Y) = 0,$$

since $h^* = \gamma^*$. Hence, c is a constant and $v(c) = \rho + v(f)$. **Q.E.D.**

Further we have

Theorem 6.4 *Let ∇^* be a torsion-free complex, H-projectively flat affine connection with symmetric and pure normalized Ricci tensor γ^* on a simply connected complex manifold* $\mathbf{M}$ *of real dimension* $2n$*. Let h be a complex analytic Riemannian metric satisfying Codazzi's equation (6.41). Then there are smooth functions ρ and λ on* $\mathbf{M}$ *satisfying the condition (6.38) such that h is given in the form (6.37).*

Proof: Let τ be an arbitrary 1-form on $\mathbf{M}$ (we can take also $\tau = 0$), and ∇ be the unique connection such that ∇^* is a complex semi-conjugate to ∇ by τ. By Theorem 5.1 we can find a complex affine immersion f with complex transversal connection form τ such that ∇^* is the connection induced by the holomorphic conormal map v of f. Now we can take any support functions ρ and λ for f and by Proposition 6.1 we get the proof of the theorem. **Q.E.D.**

Now we shall prove our second main result

Theorem 6.5 *Let ∇^* be a torsion-free complex, H-projectively flat affine connection with symmetric and pure normalized Ricci tensor γ^* on a simply connected complex manifold* $\mathbf{M}$ *of real dimension* $2n$*. Let τ be a 1-form on* $\mathbf{M}$*.*

Then for any two smooth functions ρ and λ on $\mathbf{M}$ *satisfying the condition $d\lambda = d\rho \circ J$, such that the tensor h given by*

$$h(X,Y) = \rho\gamma^*(X,Y) - \lambda\gamma^*(X,JY) + Hess^*_\rho(X,Y)$$

is non-degenerate, there is a complex affine immersion $f : \mathbf{M}^{2n} \to (\mathbf{R}^{2n+2}, J)$ with complex transversal connection form τ such that ∇^* is the connection induced by the holomorphic conormal map v of f and the given functions ρ and λ are support functions. Such f is determined uniquely up to a holomorphic affine transformation on $\mathbf{C}^{n+1}$.

Proof: If h is non-degenerate by Proposition 6.2 h is a complex Riemannian metric and satisfies the Codazzi's equation (6.41). Then the tensor $\nabla^* h$ is totally pure and the proof of Proposition 3.1 implies that h is a complex analytic Riemannian metric. By the same arguments as in Theorem 6.4 we know that there is a complex affine immersion $f : \mathbf{M}^{2n} \to (\mathbf{R}^{2n+2}, J)$ such that ∇^* is the connection induced by the holomorphic conormal map v. The condition that ρ and λ are support functions follows by Proposition 6.3.

The uniqueness part of the theorem follows by Proposition 6.3, Proposition 6.1 and the uniqueness part of Theorem 5.1. **Q.E.D.**

Corollary 6.6 *If we assume $\tau = 0$ in the conditions of Theorem 6.5 then the complex affine immersion in the conclusions of Theorem 6.5 is with holomorphic equiaffine transversal vector field.*

Proof: The assertion follows immediately from Theorem 6.5 and Corollary 3.5. **Q.E.D.**

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