Harmonic spinors of Dirac operator of connection with torsion in dimension 4

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Abstract

It is shown that the existence of a parallel (-)-spinor with respect to a metric connection with totally skew-symmetric torsion requires more local restrictions than the existence of a parallel (+)-spinor. It is proved that every harmonic spinor with respect to the Dirac operator of this connection on a compact four-dimensional spin Riemannian manifold is parallel with respect to a naturally arising metric connection with totally skew-symmetric torsion and all such spaces are classified up to a conformal transformations.

Running title: Harmonic spinors

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1 Introduction

Linear connections preserving a riemannian metric and having totally skew-symmetric torsion have been recently a subject of interest mainly because of physical reasons. These connections arise in a natural way in theoretical and mathematical physics. For example, the target space of supersymmetric sigma models with Wess-Zumino term carries a geometry of a metric connection with skew-symmetric torsion [11, 20, 21] (see also [28] and references there); in the supergravity theories, the geometry of moduli space of a class of black holes is carried out by a metric connection with skew-symmetric torsion [18], the geometry of NS-5 brane solution of type II supergravity theories is generated by a metric connection with skew-symmetric torsion [30, 31, 29].

The existence of parallel spinors with respect to a metric connection with skew- symmetric torsion on a spin Riemannian manifold is of importance in string theory since parallel spinors are associated with some string solitons (BPS solitons) [29].

The existence of parallel spinor with respect to the Levi-Civita connection on a Riemannian manifold M imposses restrictions on the holonomy group. If M is simply connected the (restricted) holonomy groups are described in [19, 35] (for non-simply connected case see [26]).

On a 4-dimensional Riemannian spin manifold (M, g) the existence of parallel spinors with respect to a metric connection ∇ with skew-symmetric torsion forces the (restricted) holonomy group of ∇ to be contained in SU(2) [29]. It is shown in [22] that this is equivalent to the local existence of a HKT structure, i.e. a hyperhermitian structure parallel with respect to ∇. It is a well known consequence of the integrability theorem in [5] that the self-dual Weyl tensor must vanish if there exists a (local) hypercomplex structure (see e.g. [17] and references there). If M is compact than the holonomy of ∇ is contained in SU(2) iff M is either a Calabi-Yau manifold or a Hopf surface. This is proved in [22] relying on the results in [16, 2]. We note that there exist Hopf surfaces which do not admit any (global) hyperhermitian structure [14] but the holonomy of ∇ is contained in SU(2) [22]. These Hopf surfaces do not admit ∇- parallel spinors by Crollary 4.3 below which shows that in the non simply connected compact case the holonomy condition is not sufficient for the existence of ∇-parallel spinors.

In the present note we are interested in the space of harmonic spinors with respect to the Dirac operator naturally associated to a metric connection ∇ with skew-symmetric torsion T on a compact 4-dimensional Riemannian spin manifold (M, g). If $dT = 0$ the structure (g, ∇) is called strong and if $d^*T = 0$ it is called closed. On a spin Riemannian manifold the spinor bundle ΣM decomposes into two parts $\Sigma M = \Sigma_+ M \oplus \Sigma_- M$ of (+)-spinors and (-)-spinors. We discovered that the existence of a ∇-parallel (-)-spinor induces more (local) restrictions on the geometry than the existence of a ∇-parallel (+)-spinor (see Theorem 4.2 bellow).

Following [23] we consider the associated Weyl structure, i.e. the unique torsion-free linear connection ∇^W preserving the conformal structure and determined by $\nabla^W g = *T \otimes g$, where $*$ is the Hodge star operator. The corresponding scalar curvature k of ∇^W is called conformal scalar curvature and has conformally invariant sign [14]. We define 'conformal change' of the connection ∇, i.e. a natural transformation of ∇ induced by a conformal rescaling of the metric and show that the dimension of the space of ∇-harmonic spinors, i.e. the dimension of the kernel of the associated Dirac operator, is a conformal invariant. This allows us to apply Gauduchon gauge and to prove our main

Theorem 1.1 *Let (M, g, ∇, T) be an oriented four-dimensional compact spin Riemannian manifold equipped with a metric connection with torsion 3-form T and of non-negative conformal scalar curvature k. Then* *i)There exists a non-zero ∇-harmonic (-) spinor iff $k = 0$ and (M, g, ∇) is a conformally equivalent to a flat torus with its hyper-Kähler metrics or a coordinate quaternionic Hopf surface with its standard flat, strong and closed HKT structure and the metric is of constant scalar curvature in the conformal class of the standard locally conformally flat metric.*

ii) There exists a non-zero ∇-harmonic (+) spinor iff $k = 0$ and (M, g, ∇) is conformally equivalent either to one of the spaces listed in i) or to a $K3$-surface with its hyper-Kähler metrics.

iii) There exists a ∇-harmonic spinor of generic type iff there exists ∇-harmonic (-) spinor.

The proof relies on the Lichnerowicz formula for Dirac operator of general connection with torsion proved in [8] (see also [1]).

If M is not compact the existence of a ∇-parallel $(+)$-spinor does not imply the vanishing of the Ricci tensor of ∇ as it could be seen in the examples in the end of the note. However, Theorem 1.1 shows that in the compact case the existence of a parallel spinor forces the Ricci tensor to be zero.

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2 Preliminaries

Let (M, g) be an n-dimensional $(n > 1)$ Riemannian manifold. Let ∇ be a linear connection. We denote by T the torsion of ∇, $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$. The connection ∇ has skew-symmetric torsion if $T(X, Y, Z) := g(T(X, Y), Z)$ is a 3-form. In the sequel, we shall call a Riemannian manifold (M, g, ∇, T) with a linear connection of skew-symmetric torsion a Riemannian manifold with torsion (briefly RT). If the torsion is closed then the manifold is called a strong RT manifold.

We note that in the physics literature the geometry of a metric connection with torsion is often called a Riemann-Cartan geometry. We prefer here the name RT manifold to emphasize on the skew-symmetric property of the torsion.

Any metric connection ∇ with skew-symmetric torsion T is connected to the Levi-Civita connection ∇^g of the metric g by

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} T(X, Y, Z), \tag{2.1}$$

Let $R = [\nabla, \nabla] - \nabla_{[,]}$ be the curvature tensor of type (1,3) of ∇. We denote the curvature tensor of type (0,4) $R(X, Y, Z, V) = g(R(X, Y)Z, V)$ by the same letter.

The Ricci tensor Ric^∇ and the scalar curvature s^∇ are defined by

$$Ric^\nabla(X, Y) = \sum_{j,i=1}^n R(e_i, X, Y e_i), \quad s^\nabla = \sum_{j,i=1}^n R(e_j, e_i, e_i, e_j).$$

Here and henceforth $e_1, e_2, ..., e_n$ is an orthonormal basis of the tangential space.

By (2.1) the Ricci tensor Ric and the scalar curvature s of the Levi-Civita connection are connected to Ric^∇ and s^∇ as follows

$$Ric(X, Y) = Ric^\nabla(X, Y) + \frac{1}{2} d^* T(X, Y) + \frac{1}{4} \sum_{i=1}^n g(T(X, e_i), T(Y, e_i)), \quad s = s^\nabla + \frac{3}{2} |T|^2, \tag{2.2}$$

where $|.|^2$ denotes the norm of a p-form.

Remark 1. It is well known (see e.g. [33]) that if ∇ is flat, i.e. with zero curvature, then (M, g, ∇, T) carries a structure of a compact Lie group, g is a biinvariant metric and ∇ is the canonical invariant connection with torsion given by the Lie bracket, except for some structures on seven dimensional spheres.

We recall the notion of KT and HKT structures. A Hermitian manifold (M, g, J) with a complex structure J and a hermitian metric g $(g(J., J.) = g(.,.))$ is called Käheler manifold with torsion (KT-manifold) if there exists a linear connection with totally skew-symmetric torsion preserving the given hermitian structure (g, J). For any hermitian structure there exists a unique connection with these properties, called Bismut connection [8, 15], (see also [36] as a general reference for connections on complex manifold). If $\Omega(.,.) = g(., J.)$ is the Kähler form then the torsion of the Bismut connection is equal to $-d\Omega(J., J., J.)$. The Lee form θ of a given hermitian structure is a 1-form defined by $\theta = d^*\Omega \circ J$. A Hyperhermitian structure on a 4n-dimensional manifold M is a triple $(J_\alpha), \alpha = 1,2,3$, of complex structures satisfying the quaternionic identities and a Riemannian metric g which is hermitian with respect to each complex structure $J_\alpha, \alpha = 1,2,3$. A Hyperhermitian manifold with torsion (briefly HKT) is a Hyperhermitian manifold equipped with a linear connection ∇ with totally skew-symmetric torsion which preserves the metric and each of the complex structures $J_\alpha, \alpha = 1,2,3$. It turns out that a Hyperhermitian manifold is HKT if and only if the Bismut connections of the three complex structures $J_\alpha, \alpha = 1,2,3$ coincide [21, 18, 15]. In particular the three Lee forms also coincide [17, 23] and the common Lee form θ is called the Lee form of the HKT structure. If each $J_\alpha, \alpha = 1,2,3$ is parallel with respect to the Levi-Civita connection then the structure is Hyper Kähler. In dimension four every hyperhermitian structure is a HKT structure with Bismut connection of torsion $T = -*\theta$ [17]. If the Lee form is closed then the HKT structure is closed since $d^*T = -*d\theta$.

Let (M, g, J) be a spin manifold with spinor bundle ΣM and let $\mu : T^*M \otimes \Sigma M \longrightarrow \Sigma M$ be the Clifford multiplication (see [25] as a general reference on spinors). Identifying T^*M and TM via the metric we shall also consider μ as a map from $TM \otimes \Sigma M$ into ΣM. We shall write the Clifford multiplication by juxtaposition, i.e.

$$\mu(\alpha \otimes \psi) = \alpha\psi, \qquad \mu(X \otimes \psi) = X\psi, \qquad \alpha \in T^*M, \quad X \in TM, \quad \psi \in \Sigma M.$$

The volume form ω_g of the metric g acts on ΣM as an involution with (-1)-eigenspace $\Sigma_+ M$ and (+1)-eigenspace $\Sigma_- M$ and ΣM decomposes as

$$\Sigma M = \Sigma_+ M \oplus \Sigma_- M, \quad \Sigma_\pm M = \{\psi | \omega\psi = \mp\psi\}.$$

Any metric connection ∇ with skew-symmetric torsion in the tangent bundle gives rise to a metric connection in the spinor bundle and it is an easy consequence of (2.1) that

$$\nabla_X \psi = \nabla^g_X \psi + \frac{1}{4}\sum_{i<j}^{n} T(X, e_i, e_j) e_i.e_j\psi, \quad X \in \Gamma(TM), \psi \in \Gamma(\Sigma M). \tag{2.3}$$

The Dirac operator D of ∇ is defined by $D = \mu \circ \nabla$ or equivalently $D\psi = \sum_{i=1}^n e_i.\nabla_{e_i}\psi$, $\psi \in \Gamma(\Sigma M)$. As a consequence of (2.3) we get the formula

$$D = D^g + \frac{3}{4}\sum_{i<j<k}^{n} T(e_i, e_j, e_k) e_i.e_j.e_k \tag{2.4}$$

where D^g is the Dirac operator of the Levi-Civita connection. From (2.4) it follows that D is self-adjoint.

The Lichnerowicz formula for D, proved in [8] states that

$$D^2 = (\nabla^3)^*\nabla^3 + \frac{1}{4}s + \frac{3}{4}dT - \frac{9}{8}|T|^2, \tag{2.5}$$

where the connection ∇^3 is a metric connection on TM with torsion $3T$ and its action on ΣM is given by

$$\nabla_X\psi = \nabla^g_X\psi + \frac{3}{4}\sum_{i<j}^{n} T(X,e_i,e_j)e_i.e_j \tag{2.6}$$

Let $\nabla^{1/3}$ be a metric connection on TM with torsion $\frac{1}{3}T$. The action of the corresponding Dirac operator $D^{1/3}$ on ΣM is

$$D^{1/3} = D^g + \frac{1}{4}\sum_{i<j<k}^{n} T(e_i,e_j,e_k)e_i.e_j.e_k \tag{2.7}$$

The operator $D^{1/3}$ is self-adjoint and Lichnerowicz formula (2.5) gives

$$(D^{1/3})^2 = \nabla^*\nabla + \frac{1}{4}s + \frac{1}{4}dT - \frac{1}{8}|T|^2, \tag{2.8}$$

It is shown in [8] that in case of complex manifold the operator $D^{1/3}$ coincides with the Dolbeault operator $\sqrt{2}(\bar\partial + \bar\partial^*)$ acting on $(0,p)$-forms provided the torsion T is the torsion of the unique Bismut connection determined by any hermitian structure.

3 Conformal transformations of RT manifolds in dimension four

From now on (M,g,∇,T) will stand for a 4-dimensional RT manifold with torsion 3-form T.

Let $*$-be the Hodge star operator. We define the torsion 1-form t as the corresponding to T 1-form via $*$-operator i.e.

$$T = -*t \tag{3.9}$$

Let $\widetilde{g} = e^f g$ be a conformal rescaling of the metric. In general, in higher dimensions there is no natural transformation of the connection ∇ induced by the conformal rescaling of the metric. However, in case of a QKT structure, i.e. when there is a quaternionic structure parallel with respect to ∇ and the torsion is (1,2)+(2,1)-form with respect to any compatible local almost complex structure (see e.g. [28]), a natural 'conformal transformation' of the connection is defined in [23]. It is well known that every 4-dimensional oriented Riemannian manifold carries a hermitian quaternionic structure (see e.g. [7]) and therefore every 4-dimensional RT manifold is a QKT space in a natural way. Following [23] we define the conformally transformed connection $\widetilde{\nabla}$ by the equation

$$\widetilde{g}(\widetilde{\nabla}_X Y, Z) = \widetilde{g}(\widetilde{\nabla}^{\widetilde{g}}_X Y, Z) - \frac{1}{2} *_{\widetilde{g}} (t+df)(X,Y,Z),$$

where $\widetilde{\nabla}^{\widetilde{g}}$ and $*_{\widetilde{g}}$ are the Levi-Civita connection and Hodge $*$-operator of $\widetilde{g}$, i.e. the torsion 1-form $\tilde{t}$ of $\tilde{\nabla}$ is given by

$$\tilde{t} = t + df. \tag{3.10}$$

We shall call (M,g,∇) and $(M,\widetilde{g},\widetilde{\nabla})$ conformally related RT manifolds.

To find distinguished RT structures on a compact 4-manifold we shall use the Gauduchon theorem for the existence of a Gauduchon metric on a compact hermitian or Weyl manifold [12, 13, 14]. This theorem can be formulated in our notations as follows: To a given 4-dimensional compact RT manifold (M,g,∇,T) there always exists a unique (up to homothety) conformally related RT manifold (M,g_G,∇_G,T_G) such that the corresponding torsion 1-form t_G is co-closed with recpect to g_G. The key point is that the torsion 1-form transforms under conformal rescaling according to (3.10) (see [34], Appendix 1). Application of this theorem leads to the following partial result of [23]

Proposition 3.1 *Let (M,g,∇,T) be a compact 4-dimensional RT manifold. In the conformal class of g there exists a unique (up to homothety) conformal metric g_G and a unique strong RT manifold (M,g_G,∇_G,T_G) conformally related to (M,g,∇,T).*

Proof. The existence of Gauduchon metric g_G means $d^*t_G=0$ which is equivalent to $dT_G=0$ by (3.9). **Q.E.D.**

On a given RT manifold we consider the associated Weyl connection ∇^W which is torsion-free and $\nabla^W g=-t\otimes g$. We note that these two conditions determine ∇^W completely (see [6] as a nice overview of Weyl structures). The conformal scalar curvature k, ie the scalar curvature of ∇^W is expressed in terms of s and the torsion 1-form t by [14, 6]

$$k=s-\frac{3}{2}|t|^2-3d^*t \tag{3.11}$$

and is a conformal invariant of weight -1 i.e. if $\widetilde{g}=e^f g$ then

$$\widetilde{k}=e^{-f}k \tag{3.12}$$

Using (2.2) we find from (3.11) $k=s^{\nabla}-3d^*t$. Thus, $k=s^{\nabla}$ if and only if the RT structure is strong.

4 Dirac operator on 4-dimensional RT manifold

From now on (M,g,∇,T) will stand for a 4-dimensional spin RT manifold.

We shall abbreviate (+)-spinors (resp. (−)-spinors) by $_+$ (resp. $_-$). Thus $\psi_\pm$ means $\psi_\pm\in\Gamma(\Sigma_\pm M)$. Direct computations using (3.9) yield

$$\sum_{i<j}^{4}T(X,e_i,e_j)e_i.e_j=-\sum_{i<j}^{4}*t(X,e_i,e_j)e_i.e_j=-(X.t+t(X))\omega. \tag{4.13}$$

We apply (4.13) to (2.3), (2.4) (2.7) and Lichnerowicz formula (2.8) to get

$$\nabla_X\psi_+=\nabla^g_X\psi_++\frac{1}{4}(X.t+t(X))\psi_+,\quad \nabla_X\psi_-=\nabla^g_X\psi_--\frac{1}{4}(X.t+t(X))\psi_-, \tag{4.14}$$

$$D\psi_+=D^g\psi_+-\frac{3}{4}t\psi_+,\qquad D\psi_-=D^g\psi_-+\frac{3}{4}t\psi_-. \tag{4.15}$$

$$D^{1/3}\psi_+=D\psi_++\frac{1}{2}t\psi_+,\qquad D^{1/3}\psi_-=D\psi_--\frac{1}{2}t\psi_-. \tag{4.16}$$

$$(D^{1/3})^2=\nabla^*\nabla+\frac{1}{4}s+\frac{1}{4}d^*t\omega-\frac{1}{8}|T|^2, \tag{4.17}$$

since $dT=d^*t\omega$ by (3.9).

Recall (cf. [19]) that under a conformal change of the metric $\widetilde{g} = e^f g$ of a spin manifold (M, g) there is an identification $\widetilde{\ }$ of the spinor bundles $\Sigma_\pm M$ of (M, g) and the spinor bundle $\widetilde{\Sigma_\pm M}$ of $(M, \widetilde{g})$ such that

$$\widetilde{X\psi} = \widetilde{X}\widetilde{\psi}, \quad \widetilde{\alpha\psi} = \widetilde{\alpha}\widetilde{\psi}, \quad \widetilde{\nabla}^{\widetilde{g}}_{\widetilde{X}}\widetilde{\psi} = \widetilde{\nabla^g_X\psi} - \frac{1}{4}\widetilde{Xdf\psi} - \frac{1}{4}\widetilde{Xf.\psi}, \tag{4.18}$$

$$\widetilde{D^{\widetilde{g}}}\widetilde{\psi} = e^{-\frac{f}{2}}\widetilde{D^g\psi} + \frac{3}{4}\widetilde{df\psi}, \tag{4.19}$$

where $\psi \in \Gamma(\Sigma M)$, $\widetilde{\psi} \in \Gamma(\widetilde{\Sigma}M)$ is the spinor corresponding to ψ, $X \in TM$, $\widetilde{X} = e^{-\frac{f}{2}}X$, $\alpha \in T^*M$, $\widetilde{\alpha} = e^{\frac{f}{2}}\alpha$.

Remark 2. It follows from (4.14) and (4.18) that two conformally related RT structures (g, ∇) and $(\widetilde{g}, \widetilde{\nabla})$ have common parallel (+)-spinors, while a ∇-parallel (-)-spinor is never $\widetilde{\nabla}$-parallel provided the function f is not a constant.

Proposition 4.1 *Let (M, g, ∇) be a 4-dimensional RT manifold. The dimension of the kernel of the Dirac operator D of ∇ is a conformal invariant.*

Proof: Since $D(\Gamma(\Sigma_\pm M)) \subset \Gamma(\Sigma_\mp M)$, it is enough to show that $dim\, Ker\, D|_{\Sigma_\pm M}$ is a conformal invariant. The torsion 1-form $\widetilde{t}$ of the structure $(\widetilde{g}, \widetilde{\nabla})$ is $t + df$ by (3.10). Then (4.15), (4.18) and (4.19) yield $\widetilde{D}\widetilde{\psi_+} = e^{-\frac{f}{2}}\widetilde{D\psi_+}$, $\quad \widetilde{D}(e^{\frac{-3}{2}f}\widetilde{\psi_-}) = e^{-2f}\widetilde{D\psi_-}$, and these equalities prove Proposition 4.1. **Q.E.D.**

In the following we shall denote by $< \cdot, \cdot >$ and $|\cdot|$ pointwise inner products and norms and by $(\cdot, \cdot)$ and $\|\cdot\|$ – the global ones, respectively. We abreviate by α^+ (resp. α^-). the self-dual (resp. anti-self-dual) part of a 2-form α with respect to the Hodge $*$-operator.

As we have already mentioned in the introduction the existence of a parallel spinor on a RT manifold implies that the holonomy of ∇ is contained in SU(2) [29]. In the next local observation we show that the existence of a parallel (-)-spinor requires the RT manifold to be strong, closed and flat. We have

Theorem 4.2 *Let (M, g, ∇, T) be an oriented four-dimensional spin RT manifold admitting a non-zero parallel spinor. Then the holonomy of ∇ is contained in SU(2) and therefore, there (locally) exists a hypercomplex structure $(J_a), a = 1, 2, 3$ such that $(g, (J_a), \nabla)$ is a HKT structure. In particular g is anti-self-dual, $k = 0$ and $dt^+ = 0$. In this case there always exists a parallel (+)-spinor.*

i) If there exists a parallel (-)-spinor then in addition $dT = 0$, $\quad d^*T = 0$, ∇ is flat and therefore it is a 4-dimensional group manifold with its standard strong, closed and flat HKT structure which is lovally conformally hyperkähler. In this case there always exists a parallel (-)-spinor.

ii) There exists a parallel spinor of generic type iff there exists a parallel (-)-spinor.

Proof. We use some elementary facts from representation theory (see e.g. [10]. Let $Cl(4)$ be the Clifford algebra of a 4-dimensional Euclidean vector space. Then $Cl(4) \cong \mathcal{H} \oplus \mathcal{H}$, where $\mathcal{H}$ is the space of quaternions. The space of spinors $\mathcal{S}^4 = \mathcal{S}_+ \otimes \mathcal{S}_-$ is a reducible $Cl(4)$- module, considered as a representation of the group Spin(4), where $\mathcal{S}_+$ and $\mathcal{S}_-$ are irreducible. The action of group $Spin(4) \cong SU(2) \times SU(2) \cong Sp(1) \times Sp(1)$ on $S^4 \cong \mathcal{H} \oplus \mathcal{H}$ is $\{A, B\}.\{s_+, s_-\} = \{As_+, Bs_-\}$. If there is a parallel spinor then the holonomy of the connection should be a subgroup of the stabilizer of the action of $Spin(4)$. If there exists a parallel spinor of generic type the stabilizer is trivial and if there exists a parallel half-spinor then the stabilizer is SU(2)=Sp(1). Hence, ∇ is flat in the first case and its holonomy is contained in SU(2) in the second one. The latter is equivalent to the existence of a local ∇- parallel hyper almost complex structure which turns out to be a hyperhermitian structure which is HKT [22]. In this case ∇ is the corresponding Bismut connection and the Weyl connection ∇^W coincides with the Obata connection (ie the unique torsion-free linear connection preserving the given hypercomplex structure) [17] which is flat [32]. This implies $k = 0$.

Further we shall use the Lichnerowicz formula (4.17). Let $t^\#$ be the vector field dual to the torsion 1-form t. We compute using (4.15) and (4.16) that

$$(D^{1/3})^2\psi_\pm = \left(D^2 \mp \frac{1}{2}(tD - Dt) + \frac{1}{4}|t|^2\right)\psi_\pm. \tag{4.20}$$

We apply (4.15) to the general identity $D^g t + tD^g = dt + d^*t - 2\nabla^g_{t^\#}$ to get

$$tD + Dt = dt + d^*t - 2\nabla_{t^\#}, \tag{4.21}$$

since $\nabla^g_{t^\#} = \nabla_{t^\#}$ because of (4.13).

Substituting Dt from (4.21) into (4.20), we derive

$$(D^{1/3})^2\psi_\pm = \left(D^2 \mp tD \pm \frac{1}{2}(dt + d^*t) \mp \nabla_{t^\#} + \frac{1}{4}|t|^2\right)\psi_\pm. \tag{4.22}$$

Inserting (4.22) into the Lichnerowicz formula (4.17), we obtain by use of (3.11) that

$$\left(D^2 - tD - \nabla_{t^\#} - \nabla^*\nabla + \frac{1}{2}dt - \frac{1}{4}k\right)\psi_+ = 0, \tag{4.23}$$

$$\left(D^2 + tD + \nabla_{t^\#} - \nabla^*\nabla - \frac{1}{2}dt - \frac{1}{4}k - \frac{3}{2}d^*t\right)\psi_- = 0. \tag{4.24}$$

Using (4.14) we see that ∇, restricted to sections of Σ^+M, coincides with the Weyl connection considered in [27]. Hence, as proved in [27], $dt^+ = 0$ and a parallel spinor in Σ^+M with respect to ∇ gives rise to a hyper-Hermitian structure $(J_a), a = 1, 2, 3$ on M and vice versa.

This can be briefly explained in our notations as follows. Any ∇-parallel (+)-spinor ψ_+ generates an almost complex structure J by $JX.\psi_+ := \sqrt{-1}X.\psi_+$ which is also ∇-parallel. In dimension 4 any almost complex structure which is parallel with respect to a metric connection with totally skew-symmetric torsion is integrable ie it is a complex structure. This is a consequence of the general fact that any 3-form is of type (1,2)+(2,1) with respect to any almost complex structure in dimension 4. In particular, the torsion of ∇ is of this type with respect to J which is equivalent to the vanishing of the Nijenhaus tensor N [22] since $g(N(.,.),.) = T^{0,3+3,0} = 0$, where $T^{0,3+3,0}$ is the (0,3)+(3,0)-part of T with respect to J. Multiplying (4.23) by ψ_+ and looking at the real and imaginary parts of the obtained equality we get $k = 0$ and $dt\psi_+ = 0$. The latter spinor equality implies $dt_+ = 0$ because of the following chain of equalities $dt^\pm.dt^\pm.\psi_+ = dt^\pm \wedge dt^\pm - |dt^\pm|^2\psi_+ = \pm dt^\pm \wedge *dt^\pm\psi_+ - |dt^\pm|^2\psi_+ = \pm|dt^\pm|^2\omega\psi_+ - |dt^\pm|^2\psi_+ = (\mp|dt^\pm|^2 - |dt^\pm|^2)\,\psi_+$. To get the other complex structures we recall [4] that there is an antilinear bundle map $j : \Sigma M \longrightarrow \Sigma M$ which commutes with the Clifford multiplication, $j^2 = -id$, j is parallel with respect to any metric connection on M. The ∇-parallel spinor $\psi_+ + j\psi_+$ determines a complex structure J_1 and the triple J, J_1, JJ_1 form a ∇-parallel hyperhermitian structure. The converse is clear.

To prove i), let us assume that there exists a non-zero parallel $(-)$spinor ψ_-. Then (4.24) reduces to $2dt\psi_- - (k + 6d^*t)\psi_- = 0$ and consequently $2 < dt\psi_-, \psi_- > -(k + 6d^*t)|\psi_-|^2 = 0$. The real and imaginary part of the last identity imply $k + 6dt^* = 0, \quad dt^- = 0$. Since the holonomy of ∇ is contained in SU(2) the first part of the theorem shows that there exist a parallel (+)-spinor and therefore $dt^+ = k = 0$. Hence, $0 = d^*t = dT$, $\quad d^*T = dt = dt^+ + dt^- = 0$ and ∇ is flat because of the existence of parallel spinor of generic type. Hence, (M, g, ∇) is a 4-dimensional group manifold with the standard HKT structure. The existence of parallel spinor of generic type on 4-dimensional HKT group manifold is clear. **Q.E.D.**

Applying the classification of compact hypercomplex surfaces of [9] (see also [27]), we get

Corollary 4.3 *Let (M, g, ∇, T) be an oriented four-dimensional compact spin RT manifold.*

i)There exists a non-zero parallel (-) spinor iff (M, g, ∇, T) is a flat torus with its hyper-Kähler metrics or a coordinate quaternionic Hopf surface with its standard flat, strong and closed HKT structure and the metric is of constant scalar curvature in the conformal class of the standard locally conformally flat metric.

ii) There exists a non-zero parallel (+) spinor iff (M, g, J_a) is either one of the spaces listed in i) or a $K3$-surface with its hyper-Kähler metrics.

iii) There exists a parallel spinor of generic type iff there exists a parallel (-)-spinor.

5 Proof of Theorem 1.1

We take the global inner product in (4.23), (4.24) and (4.21) to get

$$(5.25)\qquad ||D\psi_+||^2 - (tD\psi_+, \psi_+) - (\nabla_{t\#}\psi_+, \psi_+) = ||\nabla\psi_+||^2 - \frac{1}{2}(dt\psi_+, \psi_+) + \frac{1}{4}(k\psi_+, \psi_+),$$

$$(5.26)\qquad ||D\psi_-||^2 + (tD\psi_-, \psi_-) + (\nabla_{t\#}\psi_-, \psi_-) = ||\nabla\psi_-||^2 + \frac{1}{2}(dt\psi_-, \psi_-) + \frac{((k + 6d^*t)\psi_-, \psi_-)}{4},$$

$$(5.27)\qquad (t\psi, D\psi) - (D\psi, t\psi) = (dt\psi, \psi) + (d^*t\psi, \psi) - (2\nabla_{t\#}\psi, \psi).$$

The left side of (5.27) is an imaginary complex number and therefore the real part of the right side must vanish i.e.

$$(5.28)\qquad Re(\nabla_{t\#}\psi, \psi) = (d^*t\psi, \psi).$$

Inserting (5.28) into (5.25) and (5.26), we derive

$$(5.29)\qquad ||D\psi_+||^2 - (tD\psi_+, \psi_+) = ||\nabla\psi_+||^2 - \frac{1}{2}(dt\psi_+, \psi_+) + \frac{1}{4}((k + 4d^*t)\psi_+, \psi_+) = 0,$$

$$(5.30)\qquad ||D\psi_-||^2 + (tD\psi_-, \psi_-) = ||\nabla\psi_-||^2 + \frac{1}{2}(dt\psi_-, \psi_-) + \frac{1}{4}((k + 2d^*t)\psi_-, \psi_-) = 0,$$

Let $KerD \neq 0$. It follows from (3.12), Proposition 3.1 and Proposition 4.1 that we can assume that (M, g, ∇, T) is a strong RT manifold. Let $0 \neq \psi_+ \in KerD$ (resp. $0 \neq \psi_- \in KerD$). Then, by (5.29) and (5.30), we get

$$(5.31)\qquad ||\nabla\psi_+||^2 + \frac{1}{4}(k\psi_+, \psi_+) = 0, \qquad ||\nabla\psi_-||^2 + \frac{1}{4}(k\psi_-, \psi_-) = 0,$$

since $(dt\psi, \psi)$ is an imaginary complex number. Hence, $\nabla\psi_+ = 0$ (resp. $\nabla\psi_- = 0$) by (5.31) and $k \geq 0$. The Theorem 1.1 follows from Corollary 4.3. **Q.E.D.**

Example 1. The next examples, coming originally from [24] are considered in [27], and show that there exist (locally) strong non-closed and non-flat HKT structures and therefore admitting parallel (+)-spinors but not admitting parallel (-)-spinors. We take the description from [27].

Let $u : U \subset \mathcal{C}^2 \longrightarrow \mathcal{C}$ be a holomorphic function and consider the Riemannian metric $g_u = dz^1 d\bar{z}^1 + |u|^2 dz^2 d\bar{z}^2$. A hyperhermitian structure (J_1, J_2, J_3) compatible with the opposite orientation is defined by $J_1 + \sqrt{-1}J_2 = udz^1\bar{dz}^1 + \bar{u}dz^2\bar{dz}^2$, $\sqrt{2}J_3 = dz^1\bar{dz}^1 - |u|^2dz^2\bar{dz}^2$. The common Lee form is $\theta = \frac{1}{u}\frac{\partial u}{\partial \bar{z}^2}dz^2$. It is not closed provided $\frac{\partial u}{\partial z^1} \neq 0$ and it is an easy computation to show $d^*\theta = 0$. Thus, $(\bar{U}, g, (J_1, J_2, J_3))$ is a HKT manifold with respect to the Bismut connection ∇ admitting ∇-parallel (+)-spinor. By (2.2) $2Skew(Ric^{\nabla}) = d^*T = d\theta \neq 0$. Hence, the Ricci tensor is not zero and therefore there are no non-zero ∇-parallel (-)-spinor by Theorem 4.2.

More examples of this type are supplied by non-closed HKT structures with triholomorphic Killing vector field described in [17].

Example 2. We take the next example from [29]. Let $\mathcal{H}$ be the space of quaternions. Consider the metric $hds^2(\mathcal{H}), h = 1 + \frac{1}{|g|^2}, q \in \mathcal{H} - \{0\}$ and $ds^2(\mathcal{H})$ is the flat metric on $\mathcal{H}$, the 3-form $T = - * dh$, where $*$ is the Hodge $*$-operator with respect to $ds^2(\mathcal{H})$. The geometry of the RT manifolds $(\mathcal{H} - \{0\}, hds^2(\mathcal{H}), \nabla^{\pm}, \pm T)$ is a part of the NS-5-brane soliton of II A supergravity [29]. These RT structures are in fact HKT with respect to the hyperhermitian structure defined by right and left multiplication by the imaginary unit quaternions, respectively [29]. It is shown in [29] that the holonomy of $\nabla^{\pm}$ is SU(2) and therefore there are only ∇^{+}-parallel (+)-spinor and ∇^{-}-parallel (-)-spinor by Theorem 4.2.

6 Eigenvalue estimate

On a RT manifold (M, g, ∇, T) of particular importance is the Dirac operator $D^{1/3}$ of the metric connection with torsion $\frac{1}{3}T$. As we have already mentioned, in case of a complex manifold this operator coincides with the Dolbeault operator $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ acting on (0,p)-forms provided ∇ is the Bismut connection of the Hermitian structure [8]. An eigenvalue estimate for the Dolbeault operator on compact hermitian spin surface is given in [3]. We show below that the same estimate is valid on any compact RT manifold.

Proposition 6.1 *Let (M, g, ∇, T) be an oriented four dimensional compact spin RT manifold with non-negative conformal scalar curvature. Then the first eigenvalue λ of the Dirac operator $D^{1/3}$ satisfies the inequality*

$$\lambda^2 \geq \frac{1}{6}\inf{}_M s. \tag{6.32}$$

There is an equality in (2.1) if and only if there exists a ∇-parallel spinor in $\Sigma_+ M$ (hence the conformal scalar curvature is identically zero), the scalar curvature is constant and M is one of the spaces listed in Theorem 1.1.

Proof. Let ϕ be an eigenspinor of $D^{1/3}$ with eigenvalue λ, i.e. $D^{1/3}\phi = \lambda\phi$. Then $\phi = \phi_+ + \phi_-$ and $(D^{1/3})^2\phi_+ = \lambda^2\phi_+$. We obtain from (2.8) using (3.11) that

$$\lambda^2|\phi_+|^2 = |\nabla\phi_+|^2 + \frac{1}{6}\left(s + \frac{k}{2}\right)|\phi_+|^2. \tag{6.33}$$

Hence, if $k \geq 0$ we get the estimate (6.32) and the equality is attended iff $\nabla\phi_+ = 0$. The assertion follows from Corollary 4.3. **Q.E.D.**

Note that for the Gauduchon metric g_G the expression (3.11) yields $s_G = k_G + \frac{3}{2}|t_G|^2$. Now, the condition $k \geq 0$ implies $s_G \geq 0$. In the Gauduchon metric (6.33) holds also for (-)-spinors. The dimension of the kernel of $D^{1/3}$ is conformally invariant by applying Proposition 4.1 for the metric connection with torsion $\frac{1}{3}T$. Hence, $KerD^{1/3} = 0$ and by (6.33) the eigenvalue $\lambda^2 > 0$ provided t is not identically zero.

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List of changes

Dear Publishing Administrator,

Thank you very much for your letter concerning our paper submitted to CQG. We agree with the referee's report and amended the paper along the lines suggested by the referee.

We suplly additional information for references which were cited but were not avialable. We also include 7 additional references, namely [4], [6], [7], [10], [25], [34], [36] according to the suggestion of the referee.

We also add a little bit more explanations to make the exposition more clear and self contained which we belive is closed to the spirit suggested by the referee.

The changes are as follows:

- p.2 one sentence concerning the relatoions between parallel spinors and holonomy group;
- p.4 definitions of KT structure, Bismut connection and Lee form;
- p.6 more explanations for the Gauduchon theorem and Weyl structures;
- p.8 a brief skech of the proof of the result of Moroianu making the proof of Theorem 4.2 more clear and self contained;
- p.10 more precise explanation of the positivity of the square of the first eigenvalue of the Dirac operator.

Yours sincerely,
Petar Dalakov and Stefan Ivanov