Holomorphic and Killing vector fields on compact balanced Hermitian manifolds

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Abstract

We extend the classical theorems for nonexistence of Killing and holomorphic vector fields on compact Kähler manifolds to compact balanced Hermitian manifolds. We express the obstructions to the existence of Killing and holomorphic vector fields on compact balanced Hermitian manifolds in terms of the Ricci tensors of the Levi-Civita connection as well as in terms of the Ricci tensors of the Chern connection. We show that every affine vector field with respect to the Chern connection on a compact balanced Hermitian manifold is holomorphic.

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1 Introduction

The well known vanishing theorem of Bochner says that if the Ricci tensor of a compact Riemannian manifold is nonpositive, then every Killing vector field is parallel. Moreover, if the Ricci tensor is nonpositive and negative at least at one point, then there are no Killing vector fields other than zero and the isometry group is finite.

Certainly, this theorem is true for compact Kähler manifolds. It is also well known that every Killing vector field on a compact Kähler manifold is (real) holomorphic, i.e. it is an infinitesimal automorphism of the complex structure. Conversely, every (real) holomorphic

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vector field with co-closed dual 1-form is Killing [31]. The theorem of Bochner for (real) holomorphic vector fields on compact Kähler manifolds is formulated in a similar way as for Killing vector fields. Namely, it states that if the Ricci tensor of a compact Kähler manifold is nonpositive, then every (real) holomorphic vector field is parallel. Moreover, if the Ricci tensor is nonpositive and negative at least at one point, then there are no (real) holomorphic vector fields other than zero [31] and the holomorphic automorphism group is finite [19].

However, a Killing vector field may not be (real) holomorphic on a compact Hermitian manifold. There exists a Bochner type theorem for (real) holomorphic vector fields on compact Hermitian manifolds without the statement of finiteness of the holomorphic automorphism group. This theorem is formulated in terms of the Chern connection and its mean curvature. In fact, on a compact Hermitian manifold with nonpositive mean curvature every (real) holomorphic vector field is parallel with respect to the Chern connection; if in addition the mean curvature is negative at least at one point, then there are no (real) holomorphic vector fields other than zero [22, 14, 28, 21] and the holomorphic automorphism group is discrete. We note that this is a part from the general result for the nonexistence of holomorphic sections of a holomorphic vector bundle over a compact Hermitian manifold [22, 14] (see also [28, 21]).

In the present paper we consider questions of existence of Killing vector fields and (real) holomorphic vector fields on compact balanced Hermitian manifolds.

Balanced Hermitian manifolds are Hermitian manifolds with co-closed fundamental form or equivalently with zero Lee form. They have been studied intensively in [23, 1, 2, 3]; in [14] they are called semi-Kähler manifolds of special type. This class of manifolds includes the class of Kähler manifolds but also many important classes of non-Kähler manifolds, such as: complex solvmanifolds, twistor spaces of oriented Riemannian 4-manifolds, 1-dimensional families of Kähler manifolds (see [23]), some compact Hermitian manifolds with flat Chern connection (see [14]), twistor spaces of oriented distinguished Weyl structure on compact self-dual 4-manifolds [15], twistor spaces of quaternionic Kähler manifolds [26, 4], manifolds obtained as modification of compact Kähler manifolds [1] and of compact balanced manifolds [2] (see also [3]).

On a balanced Hermitian manifold ($\mathbf{M}$,g,J) there are two Ricci tensors ρ and ρ^* associated with the Levi-Civita connection ∇ of the metric g and two Ricci tensors k and k^* associated with the canonical Chern connection D generated by the metric g and the complex structure J. We note that the $(1,1)$-form corresponding to the tensor k represents the first Chern class of $\mathbf{M}$ and the $(1,1)$-form corresponding to the tensor k^* is the mean curvature. All these Ricci tensors coincide on a Kähler manifold.

We express obstructions to the existence of Killing vector fields in terms of the Ricci tensors ρ and $\rho*$ of the Levi-Civita connection. Namely, we prove

Theorem 1.1 *Let ($\mathbf{M}$,g,J) be a compact balanced Hermitian manifold.*

i) If the tensor $\rho+\rho^*$ is nonpositive, then every Killing vector field X on $\mathbf{M}$ satisfies the conditions

$$\rho(X,X)+\rho^*(X,X)=0, \quad \nabla'\omega_X=0, \quad \delta\omega_{JX}=0,$$

where ω_X and ω_{JX} are the dual 1-forms of X and JX, respectively and $\nabla'\omega_X$ denotes the (1,1)-part of $\nabla\omega_X$.

ii) If the tensor $\rho + t\rho^*$ is nonpositive for some real $t \in [0, 1)$, then every Killing vector field X on $\mathbf{M}$ is ∇-parallel and satisfies the conditions

$$\rho(X, X) + t\rho^*(X, X) = 0, \quad \delta\omega_{JX} = 0.$$

iii) If the tensor $\rho + t\rho^*$, $t \in [0, 1]$ is nonpositive and negative at least at one point in $\mathbf{M}$, then there are no Killing vector fields on $\mathbf{M}$ other than zero and the isometry group is finite.

We find a sufficient condition for a Killing vector field to be (real) holomorphic in terms of the Chern form of the Chern connection. We also express obstructions to the existence of Killing vector fields and (real) holomorphic vector fields on a compact balanced Hermitian manifold in terms of the Chern connection. More precisely, we prove

Theorem 1.2 *Let $(\mathbf{M}, g, J)$ be a compact balanced Hermitian manifold.*
i) If the tensor k is nonpositive, then:
a) every Killing vector field X on $\mathbf{M}$ is (real) holomorphic;
b) every (real) holomorphic vector field X satisfies the conditions

$$k(X, X) = 0, \quad \delta\omega_X = \delta\omega_{JX} = 0.$$

ii) If the tensor $k + k^*$ is nonpositive, then every holomorphic vector field is D-parallel, the dual 1-form ω_X of a Killing vector field X has a holomorphic $(1, 0)$-part and

$$k(X, X) + k^*(X, X) = 0, \quad \nabla'\omega_X = 0.$$

iii) If anyone of the tensors k or $k + k^*$ is nonpositive and negative at least at one point in $\mathbf{M}$, then there are neither non-zero Killing vector fields nor non-zero holomorphic vector fields on $\mathbf{M}$ and the isometry group is finite.

Remark 1. The nonexistence of holomorphic vector fields on a compact complex manifold implies that the group of holomorphic automorphisms is discrete. The result of S.Kobayashi [19] states that if the first Chern class is negative, then the holomorphic automorphism group is finite. We do not know whether the condition $\rho + \rho^* < 0$ or $k + k^* < 0$ (the condition $k + k^* < 0$ implies $\rho + \rho^* < 0$, as we shall see below in Section 3) on a compact balanced Hermitian manifold leads to special restrictions of the group of holomorphic transformations, in particular to its finiteness.

In the last section we show that every infinitesimal automorphism of the Chern connection on a compact balanced Hermitian manifold is also an infinitesimal automorphism of the complex structure.

2 Preliminaries

Let $(\mathbf{M}, g, J)$ be a $2n$-dimensional Hermitian manifold with Riemannian metric g and complex structure J. The algebra of all C^∞ vector fields on $\mathbf{M}$ will be denoted by $\mathbf{XM}$. The Kähler form Ω of the Hermitian structure is defined by $\Omega(X, Y) = g(JX, Y)$; $\quad X, Y \in \mathbf{XM}$. The associated Lee form θ is given by $\theta = -\delta\Omega \circ J$.

We denote by ∇ and $R = [\nabla, \nabla] - \nabla_{[,]}$ the Levi-Civita connection of the metric g and the Riemannian curvature tensor, respectively. The corresponding curvature tensor of type (0,4) is given by the equality $R(X,Y,Z,V) = g(R(X,Y)Z,V); \quad X,Y,Z,V \in \mathbf{XM}$.

Further, ρ and ρ^* will stand for the Ricci tensor and *-Ricci tensor, respectively. We have

$$\rho^*(X,Y) = \sum_{j=1}^{2n} R(e_j, X, JY, Je_j), \quad X,Y \in \mathbf{XM}.$$

Here and henceforth $\{e_1, ...e_{2n}\}$ denotes an orthonormal frame.

We denote by D, T and K the canonical Chern (Hermitian) connection of the Hermitian structure (g, J), its torsion tensor and its curvature tensor (the Hermitian curvature tensor), respectively. We recall that the Chern connection D is the unique linear connection preserving the metric g and the complex structure J, so that the torsion tensor T of D has the property $T(JX,Y) = T(X,JY); \quad X,Y \in \mathbf{XM}$. This implies (e.g. [6]):

$$T(JX,Y) = JT(X,Y), \qquad X,Y \in \mathbf{XM}. \tag{2.1}$$

The corresponding torsion tensor of type (0,3) is defined by the equality

$$T(X,Y,Z) = g(T(X,Y),Z), \quad X,Y,Z \in \mathbf{XM}.$$

The curvature K of D has the following properties:

$$K(JX,JY)Z = K(X,Y)Z, \quad K(X,Y)JZ = JK(X,Y)Z, \quad X,Y,Z \in \mathbf{XM}. \tag{2.2}$$

We also note the Ricci identity for D with respect to a vector field $Z \in \mathbf{XM}$:

$$D_X D_Y Z - D_Y D_X Z = K(X,Y)Z - D_{T(X,Y)}Z, \qquad X,Y \in \mathbf{XM}. \tag{2.3}$$

The two connections ∇ and D are related by the following identity

$$g(\nabla_X Y, Z) = g(D_X Y, Z) + \frac{1}{2} d\Omega(JX,Y,Z), \qquad X,Y,Z \in \mathbf{XM}. \tag{2.4}$$

It follows from (2.4) that

$$T(X,Y,Z) = -\frac{1}{2} d\Omega(JX,Y,Z) - \frac{1}{2} d\Omega(X,JY,Z), \qquad X,Y,Z \in \mathbf{XM}. \tag{2.5}$$

There are three Ricci-type tensors k, k^* and s associated with the curvature tensor K defined by

$$k(X,Y) = -\frac{1}{2} \sum_{j=1}^{2n} (K(X,JY,e_j,Je_j), \quad k^*(X,Y) = -\frac{1}{2} \sum_{j=1}^{2n} K(e_j,Je_j,X,JY),$$

$$s(X,Y) = \sum_{j=1}^{2n} K(e_j,X,Y,e_j), \quad X,Y \in \mathbf{XM}.$$

The $(1,1)$-form κ corresponding to the tensor k represents the first Chern class of $\mathbf{M}$ (further we shall call it the Chern form) and the (1,1)-form κ^* corresponding to the tensor k^* is the mean curvature of the holomorphic tangent bundle $\mathbf{T}^{(1,0)}\mathbf{M}$ with the hermitian metric induced by g.

For an arbitrary vector field X we denote by ω_X its dual 1-form defined by $\omega_X(Y) = g(X,Y), \quad Y \in \mathbf{XM}$. It follows from (2.4) that

$$\delta\omega_X = -\sum_{i=1}^{2n}(D_{e_i}\omega_X)e_i - \theta(X). \tag{2.6}$$

3 Balanced Hermitian manifolds

We recall the definition of a balanced Hermitian manifold and some equivalent conditions given in [14, 23], for completeness:

Definition: *A balanced* Hermitian manifold (**M**,g,J) is a Hermitian manifold of complex dimension n which satisfies one of the following equivalent conditions:

i) $\delta\Omega = 0, (\theta = 0)$; ii) $d\Omega^{n-1} = 0$;

iii) $\Delta_\partial f = \Delta_{\bar\partial} f = \frac{1}{2}\Delta_d f$ for any smooth function f on $\mathbf{M}$, where $\Delta_\partial, \Delta_{\bar\partial}$, and Δ_d denote the Laplacians with respect to the operators $\partial, \bar\partial$, and d, respectively.

We shall use local holomorphic coordinates $\{z^\alpha\}, \alpha = 1, ..., n$ and the corresponding frame field

$$\{\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^{\bar\alpha}} = \overline{\frac{\partial}{\partial z^\alpha}}\}, \quad \alpha = 1, ..., n; \quad \bar\alpha = \bar 1, ..., \bar n,$$

for some calculations. With respect to local holomorphic coordinates the first Bianchi identity for the Hermitian curvature K can be written by virtue of (2.2) in the form

$$K_{\alpha\bar\beta\gamma\bar\lambda} - K_{\gamma\bar\beta\alpha\bar\lambda} = -D_{\bar\beta}T_{\alpha\gamma\bar\lambda}. \tag{3.7}$$

By the condition $\delta\Omega = 0$, it follows from (3.7) that (see [8])

$$s(X,Y) = s(Y,X) = k(X,Y), \quad X, Y \in \mathbf{XM}. \tag{3.8}$$

It follows from (2.6) that on a balanced Hermitian manifold we have

$$\delta\omega = -\sum_{i=1}^{2n}(D_{e_i}\omega)e_i. \tag{3.9}$$

For any (0,2)-tensor a, we denote by a^t the (0,2)-tensor defined by $a^t(X,Y) = a(Y,X)$, $X, Y \in \mathbf{XM}$. Then the symmetric part of the tensor a is given by

$$Sym(a) = \frac{1}{2}(a + a^t).$$

If a and b are tensors of type (0,2), then the induced by g scalar product of a and b will be denoted by the same letter. We adopt the usual conventions:

$$g(a,b) = \sum_{i,j=1}^{2n} a(e_i,e_j)b(e_i,e_j); \qquad g(a^t,b) = g(a,b^t) = \sum_{i,j=1}^{2n} a(e_i,e_j)b(e_j,e_i).$$

For a fixed vector field X we obtain the following (0,2)-tensors $i_X T$ and $j_X T$ from the torsion tensor T:

$$(i_X T)(Y, Z) = T(X, Y, Z); \quad (j_X T)(Y, Z) = T(Y, Z, X), \quad Y, Z \in \mathbf{XM}. \tag{3.10}$$

It follows from (2.1) and (3.10) that the tensor $i_X T$ is J-invariant, while the tensor $j_X T$ is J-antiinvariant, i.e.

$$(i_X T)(JY, JZ) = (i_X T)(Y, Z); \quad (j_X T)(JY, JZ) = -(j_X T)(Y, Z).$$

By these denotations, we have the following expressions for the (0,2) + (2,0)-parts of the Ricci tensors ρ and ρ^*:

Proposition 3.1 *On a balanced Hermitian manifold the following equalities hold*

$$\rho^*(X, Y) = \rho^*(JX, JY) = \rho^*(Y, X), \quad X, Y \in \mathbf{XM}; \tag{3.11}$$

$$\rho(X, Y) - \rho(JX, JY) = -g(i_X T, (i_Y T)^t), \quad X, Y \in \mathbf{XM}. \tag{3.12}$$

Proof. Using (2.4), we calculate $R^{\lambda}_{\alpha\beta\gamma}$ with respect to local holomorphic coordinates. From (2.2) we have $K^{\lambda}_{\alpha\beta\gamma} = 0$. Taking into account $d^2\Omega = 0$ and (2.5), we obtain

$$R^{\lambda}_{\alpha\beta\gamma} = -\frac{1}{2} D_\gamma T^{\lambda}_{\alpha\beta} + \frac{1}{4} T^{\sigma}_{\beta\gamma} T^{\lambda}_{\sigma\alpha} + \frac{1}{4} T^{\sigma}_{\gamma\alpha} T^{\lambda}_{\sigma\beta}.$$

Here and further the summation convention is assumed.

Using the last equality, we find after contractions

$$\rho^*_{\beta\gamma} = 0, \qquad \rho_{\beta\gamma} = -\frac{1}{2} T^{\sigma}_{\beta\lambda} T^{\lambda}_{\gamma\sigma}. \tag{3.13}$$

The first equality in (3.13) implies (3.11). The second equality in (3.13) and (3.10) imply (3.12). **Q.E.D.**

The connections between the (1,1)-parts of the Ricci tensors are clarified by the next

Proposition 3.2 *The Ricci tensors on a balanced Hermitian manifold satisfy the following identities:*

$$k(X, Y) - \rho^*(X, Y) = \frac{1}{4} g(j_X T, j_Y T); \tag{3.14}$$

$$k(X, Y) + k^*(X, Y) - \frac{1}{2}\left(\rho(X, Y) + \rho(JX, JY)\right) - \rho^*(X, Y) = \frac{1}{2} g(i_X T, i_Y T); \tag{3.15}$$

$$k(X, X) + k^*(X, X) - \rho(X, X) - \rho^*(X, X) = \|Sym(i_X T)\|^2, \tag{3.16}$$

where $X, Y \in \mathbf{XM}$ *and* $\|.\|^2$ *is the usual tensor norm.*

Proof. Using (2.4), we calculate the difference $R_{\alpha\bar{\beta}\gamma\bar{\lambda}} - K_{\alpha\bar{\beta}\gamma\bar{\lambda}}$ with respect to local holomorphic coordinates. Applying two contractions to the obtained equality and taking into account (2.5) and (3.7), we obtain (3.14) and (3.15). The identity (3.16) follows immediately from (3.15) and (3.12). **Q.E.D.**

Let η be an 1-form. We denote by $d'\eta, D'\eta$ and $\nabla'\eta$ the (1,1)-part (with respect to the complex structure J) of the exterior derivative $d\eta$, the covariant derivative $D\eta$ with respect to the Chern connection and the covariant derivative $\nabla\eta$ with respect to the Levi-Civita connection of η, respectively; for the ((0,2)+(2,0))-parts of $d\eta, D\eta$ and $\nabla\eta$ we use the denotations $d''\eta, D''\eta$ and $\nabla''\eta$, respectively. For example,

$$d'\omega_X(Y,Z) = \frac{1}{2}\left(d\omega_X(Y,Z) + d\omega_X(JY,JZ)\right); \quad d''\omega_X(Y,Z) = \frac{1}{2}\left(d\omega_X(Y,Z) - d\omega_X(JY,JZ)\right)$$

where $Y, Z \in \mathbf{XM}$.

The next integral formulas are essential to the proof of our main results.

Proposition 3.3 *Let* $(\mathbf{M}, g, J)$ *be a compact balanced Hermitian manifold. Then, for every smooth vector field* X *on* $\mathbf{M}$*, we have*

$$(3.17) \int_{\mathbf{M}} 2\|Sym(D''\omega_X)\|^2\, dV = \int_{\mathbf{M}} \left(\|D''\omega_X\|^2 - k(X,X) + \frac{1}{2}(\delta\omega_X)^2 + \frac{1}{2}(\delta\omega_{JX})^2\right) dV,$$

$$(3.18) \qquad \int_{\mathbf{M}} 2\|Sym(D''\omega_X\|^2\, dV =$$

$$= \int_{\mathbf{M}} \left(\|D'\omega_X\|^2 - k(X,X) - k^*(X,X) + \frac{1}{2}(\delta\omega_X)^2 + \frac{1}{2}(\delta\omega_{JX})^2\right) dV,$$

$$(3.19) \qquad \int_{\mathbf{M}} \left(g(i_X T, D'\omega_X) - g(j_X T, D''\omega_X)\right) dV = \int_{\mathbf{M}} \left(k(X,X) - k^*(X,X)\right) dV.$$

Proof. We consider the following real 1-form

$$\varphi = X^\alpha D_\alpha X_\beta dz^\beta + X^{\bar{\alpha}} D_{\bar{\alpha}} X_{\bar{\beta}} dz^{\bar{\beta}}$$

and compute its co-differential $\delta\varphi$. Using (3.9) and taking into account (2.3), (2.1), (2.2) and (3.8), we find

$$(3.20) \qquad -\delta\varphi = g(D''\omega_X, (D''\omega_X)^t) + k(X,X) - \frac{1}{2}X(\delta\omega_X) - \frac{1}{2}JX(\delta\omega_{JX}).$$

Integrating (3.20) over $\mathbf{M}$ and using the identity

$$2\|Sym(D''\omega_X)\|^2 = \|D''\omega_X\|^2 + g(D''\omega_X, (D''\omega_X)^t),$$

we obtain (3.17). To prove (3.18), we consider the following real 1-forms

$$\psi = X^\beta D_\alpha X_\beta dz^\alpha + X^{\bar{\beta}} D_{\bar{\alpha}} X_{\bar{\beta}} dz^{\bar{\alpha}},$$

$$\eta = X^{\bar{\beta}} D_\alpha X_{\bar{\beta}} dz^\alpha + X^\beta D_{\bar{\alpha}} X_\beta dz^{\bar{\alpha}}.$$

Using (3.9) and taking into account (2.3), (2.1) and (2.2), we find

$$-\delta\psi - \delta\eta = \|D''\omega_X\|^2 - \|D'\omega_X\|^2 + k^*(X,X).$$

Integrating the last equality over $\mathbf{M}$, we obtain

$$\int_{\mathbf{M}} \left(\|D''\omega_X\|^2 + k^*(X,X)\right) dV = \int_{\mathbf{M}} \|D'\omega_X\|^2 \, dV. \tag{3.21}$$

Now, (3.18) follows from (3.17) and (3.21).

Further, we consider the real 1-form

$$\varphi^* = T_{\sigma\alpha\bar\beta} X^\alpha X^{\bar\beta} dz^\sigma + T_{\bar\sigma\bar\alpha\beta} X^{\bar\alpha} X^\beta dz^{\bar\sigma}.$$

We calculate $\delta\varphi^*$ using (3.9), (3.7) and (3.8). Integrating the obtained equality over $\mathbf{M}$, we get (3.19). **Q.E.D.**

4 Proof of the theorems

We recall that a real vector field X is said to be Killing (resp. real holomorphic) if $L_X g = 0$ (resp. $L_X J = 0$), where L_X denotes the Lie derivative with respect to X. In terms of the Levi-Civita connection, the Killing condition is expressed by the equalities

$$Sym(\nabla''\omega_X) = 0, \qquad Sym(\nabla'\omega_X) = 0. \tag{4.22}$$

The second condition of (4.22) implies $\delta\omega_X = 0$.

Using (2.4), we express the Killing condition in terms of the Chern connection as follows:

$$Sym(D''\omega_X) = 0, \tag{4.23}$$

$$Sym(D'\omega_X) = -Sym(i_X T). \tag{4.24}$$

A real vector field X is (real) holomorphic iff the following condition holds (see e.g. [9])

$$D"\omega_X = 0, \tag{4.25}$$

i.e. the (1,0)-part of X is a holomorphic vector field.

4.1 Proof of Theorem 1.2

i) a) Let X be a Killing vector field. Then, the Killing conditions (4.23), (4.24) and the formula (3.17) imply

$$\int_{\mathbf{M}} \left(\|D''\omega_X\|^2 - k(X,X) + \frac{1}{2}(\delta\omega_{JX})^2\right) dV = 0. \tag{4.26}$$

If $k(X,X) \leq 0$, then (4.26) implies $k(X,X) = 0, \quad \delta\omega_{JX} = 0, \quad D''\omega_X = 0$. Hence, the Killing vector field X is (real) holomorphic.

b) For a (real) holomorphic vector field X, we get from (3.17) and (4.25) that

$$\int_{\mathbf{M}} \left(-k(X,X) + \frac{1}{2}(\delta\omega_X)^2 + \frac{1}{2}(\delta\omega_{JX})^2\right) dV = 0.$$

If $k(X,X) \leq 0$, then the assertion follows immediately from the last equality .

ii) For a Killing vector field X we find from (3.18) that

$$\int_{\mathbf{M}} \left(\|D'\omega_X\|^2 - k(X,X) - k^*(X,X) + \frac{1}{2}(\delta\omega_{JX})^2\right) dV = 0. \tag{4.27}$$

If $k(X,X) + k^*(X,X) \leq 0$, then (4.27) implies $k(X,X) + k^*(X,X) = 0$, $\quad \delta\omega_{JX} = 0$ and $D'\omega_X = 0$. The last equality means that the (1,0)-part of ω_X is holomorphic and $d'\omega_X = 0$. Combining the last equality with the second equality of (4.22), we obtain $\nabla'\omega_X = 0$.

If X is a (real) holomorphic vector field, we get from (3.18) and (4.25) that

$$\int_{\mathbf{M}} \left(\|D'\omega_X\|^2 - k(X,X) - k^*(X,X) + \frac{1}{2}(\delta\omega_x)^2 + \frac{1}{2}(\delta\omega_{JX})^2\right) dV = 0.$$

If $k(X,X) + k^*(X,X) \leq 0$ then the last formula implies in particular that $D'\omega_x = 0$. This equality and (4.25) imply that the (real) holomorphic vector field X is D-parallel.

The last part iii) of Theorem 1.2 follows directly from the formulas (4.26), (4.27) and the well known theorem for compactness of the isometry group of a compact Riemannian manifold (see e.g. [20], Ch. 2). **Q.E.D.**

4.2 Proof of Theorem 1.1

i) Let X be a vector field in $\mathbf{XM}$. Using (2.4), we find the following equality

$$4\|D'\omega_X\|^2 = \|d'\omega_X\|^2 + 4\|Sym(D'\omega_X)\|^2. \tag{4.28}$$

If X is a Killing vector field, then (4.28), (4.24) and the second equality of (4.22) imply

$$\|D'\omega_X\|^2 = \|\nabla'\omega_X\|^2 + \|Sym(i_XT)\|^2.$$

Substituting the last equality into (4.27), we obtain

$$\int_{\mathbf{M}} \left(\|\nabla'\omega_X\|^2 + \|Sym(i_XT)\|^2 - k(X,X) - k^*(X,X) + \frac{1}{2}(\delta\omega_{JX})^2\right) dV = 0. \tag{4.29}$$

Using (3.16), we get from (4.29) that

$$\int_{\mathbf{M}} \left(\|\nabla'\omega_X\|^2 - \rho(X,X) - \rho^*(X,X) + \frac{1}{2}(\delta\omega_{JX})^2\right) dV = 0. \tag{4.30}$$

The equality (4.30) implies the conclusions in i) at once.

We shall prove ii) and iii) simultaneously. If X is a Killing vector field, we take the scalar product of (4.24) and $D'\omega_X$ and find

$$\|D'\omega_X\|^2 + g(D'\omega_X, (D'\omega_X)^t) + g(i_XT, D'\omega_X) + g(i_XT, (D'\omega_X)^t) = 0. \tag{4.31}$$

Further, we consider the following real 1-form

$$\xi = X^{\bar{\alpha}} D_{\bar{\alpha}} X_\beta dz^\beta + X^\alpha D_\alpha X_{\bar{\beta}} dz^{\bar{\beta}}.$$

Using (2.3), (2.2) and (2.1), we get

$$-\delta\xi = g(D'\omega_X, (D'\omega_X)^t) - g(i_X T, (D'\omega_X)^t) + \frac{1}{2} JX(\delta\omega_{JX}),$$

since $\delta\omega_X = 0$ for a Killing vector field.

Integrating the last equality over $\mathbf{M}$ and taking into account (4.31), we obtain

$$\int_{\mathbf{M}} \left(\|D'\omega_X\|^2 + \frac{1}{2}(\delta\omega_{JX})^2 + g(i_X T, D'\omega_X) \right) dV = 0. \tag{4.32}$$

On the other hand, from (2.4) we have

$$\nabla''\omega_X = D''\omega_X + \frac{1}{2} j_X T.$$

By using the last equality and (3.14) we find

$$\int_{\mathbf{M}} \|\nabla''\omega_X\|^2 \, dV = \int_{\mathbf{M}} \left(\|D''\omega_X\|^2 + g(j_X T, D''\omega_X) + k(X,X) - \rho^*(X,X) \right) dV. \tag{4.33}$$

Finally, we get from (4.33), (3.21), (4.32) and (3.19) the equality

$$\int_{\mathbf{M}} \left(\|\nabla''\omega_X\|^2 + \rho^*(X,X) - \frac{1}{2}(\delta\omega_{JX})^2 \right) dV = 0. \tag{4.34}$$

For an arbitrary real number $t \in (0,1)$, the equalities (4.30) and (4.34) imply

$$\int_{\mathbf{M}} \left(\|\nabla'\omega_X\|^2 + (1-t)\|\nabla''\omega_X\|^2 - (\rho(X,X) + t\rho^*(X,X)) + \frac{t}{2}(\delta\omega_{JX})^2 \right) dV = 0. \tag{4.35}$$

Now, the assertions of ii) and iii) follow in a straightforward way from (4.35), (4.34), (4.30) and the theorem for compactness of the isometry group of a compact Riemannian manifold (see e.g. [20], Ch. 2). **Q.E.D.**

Example 1. Let $\mathbf{M} = G/\Gamma$ be a compact quotient of a complex Lie group G with respect to its discrete subgroup Γ. It is well known that every compact Hermitian manifold with flat Chern connection is isomorphic with $\mathbf{M}$ endowed with its flat hermitian structure [16]. If the group G is two-step nilpotent, then $\mathbf{M}$ is a non-Kähler balanced Hermitian manifold (a typical example is the complex Iwasawa manifold which is obtained by taking the 3-dimensional complex Heisenberg group for G and its discrete subgroup consisting of the matrices whose entries are Gaussian integers for Γ). Since $k = k^* = 0$, then every Killing vector field is (real) holomorphic and hence it is parallel with respect to the flat Chern connection according to Theorem 1.2.

Example 2. Let $(\mathbf{M}^{4n}, g)$ be a compact $4n$-dimensional quaternionic Kähler manifold with constant negative scalar curvature s for $n \geq 2$, and an oriented self-dual Einstein Riemannian

4-manifold of constant negative scalar curvature s for $n = 1$. The twistor space of $\mathbf{M}^{4n}$ is a 2-sphere bundle $\mathbf{Z}$ over $\mathbf{M}^{4n}$ whose fibre at any point $p \in \mathbf{M}^{4n}$ consists of all complex structures on the tangent space $T_p\mathbf{M}^{4n}$ at p which are compatible with the given structure. The $(4n+2)$-dimensional twistor space $\mathbf{Z}$ admits a complex structure J determined in [7] for $n = 1$ and in [27] for $n \geq 2$. There exists a natural 1-parameter family of hermitian metrics $h_c, c > 0$ on $(\mathbf{Z}, J)$ such that the projection $\pi : \mathbf{Z} \to \mathbf{M}^{4n}$ is a Riemannian submersion with totally geodesic fibres [13]. The twistor space $(\mathbf{Z}, J, h_c), c > 0$ is a compact balanced Hermitian manifold [15, 24, 4]. The curvature of $(\mathbf{Z}, h_c)$ for $n = 1$ is calculated by many authors [10, 11, 12, 13, 18, 29]. The Ricci tensor ρ_c and $*$-Ricci tensor ρ^*_c of $(\mathbf{Z}, h_c, J)$ for $n \geq 2$ are given in [4] by formulas (3.11) and (3.12), respectively. The latter formulas are also valid when $\mathbf{M}^4$ is an oriented self-dual Einstein Riemannian manifold. Using (3.11) and (3.12) from [4] by direct calculations we obtain the following:

i) If $sc^2 < -\frac{4(n+2)}{n}$, then $\rho_c + \rho^*_c < 0$ and Theorem 1.1 implies that there are no non-zero Killing vector fields on $(\mathbf{Z}, h_c)$;

ii) If $-\frac{4(n+2)t}{1-t} - \frac{4(n+2)}{n(1-t)}\sqrt{n((n+1)t^2-1)} \leq sc^2 \leq -\frac{4(n+2)t}{1-t} + \frac{4(n+2)}{n(1-t)}\sqrt{n((n+1)t^2-1)}$ for some real number $t \in [\frac{1}{\sqrt{n+1}}, 1)$, then $\rho_c + t\rho^*_c \leq 0$. According to Theorem 1.1 any Killing vector field on $(\mathbf{Z}, h_c)$ has to be parallel with respect to the Levi-Civita connection of h_c and tangent to the fiber of the twistor fibration. This is because $\rho_c + t\rho^*_c$ is then negative-definite on the horizontal distribution. Since the fibers are 2-spheres, this forces the Killing vector field to be identically zero.

Note that the Ricci tensor ρ_c of h_c is not non-positive by (3.11) from [4] and the Bochner vanishing theorem cannot be applied.

On the other hand we have the following:

In the case $n = 1$, every Killing vector field on the 6-dimensional manifold $(\mathbf{Z}, h_c, J)$ is (real) holomorphic since there are exactly three complex structures compatible with h_c even localy on $\mathbf{Z}$ [5]. The considerations in [17], p. 136 together with the results in [20], Ch. 4 show that any non-zero Killing vector field on $(\mathbf{Z}, h_c)$ generates a non-zero Killing vector field on $(\mathbf{M}, g)$. Hence, there are no non-zero Killing vector fields on $(\mathbf{Z}, h_c)$ since $(\mathbf{M}, g)$ is an Einstein manifold with negative scalar curvature.

In the case $n \geq 2$ every non-zero Killing vector field on $(\mathbf{Z}, h_c)$ which is simultaneously (real) holomorphic with respect to the complex structure J also generates a non-zero Killing vector field on $(\mathbf{M}, g)$ (cf. [27, 25]). Hence, there are no non-zero Killing vector fields on $(\mathbf{Z}, h_c, J)$ which are (real) holomorphic with respect to the complex structure J.

Remark 2. If the Chern form κ is negative then the first Chern class is negative. By the theorem of S.Kobayashi [20] it follows that there are no holomorphic vector fields and hence there are no Killing vector fields with respect to any Kähler metric on $\mathbf{M}$.

As a consequence of the proofs of the theorems we get

Proposition 4.1 *Every vector field which is an infinitesimal automrphism of the Chern connection (affine with respect to the Chern connection) on a compact balanced Hermitian manifold is (real) holomorphic.*

Proof. We recall that a real vector field X is said to be affine with respect to the Chern connection if $L_X D = 0$, where L_X denotes the Lie derivative. Using the general formulas expressing $L_X D$ in terms of the curvature and the torsion of D (see [30]) and taking into account (2.2) and (2.1), we find that

$$(L_X D)^{\bar\gamma}_{\alpha\beta} = D_\alpha D_\beta X^{\bar\gamma}. \tag{4.36}$$

Now, let X be an affine vector field with respect to the Chern connection. Then (4.36) implies

$$D_\alpha D_\beta X^{\bar\gamma} = 0 \tag{4.37}$$

Integrating over $\mathbf{M}$ the co-differential $\delta\psi$ of the 1-form $\psi = X^\beta D_\alpha X_\beta dz^\alpha + X^{\bar\beta} D_{\bar\alpha} X_{\bar\beta} dz^{\bar\alpha}$ and taking into account (4.37), we obtain

$$\int_{\mathbf{M}} \|D''\omega_X\|^2 \, dV = 0.$$

Hence, X is a (real) holomorphic vector field. **Q.E.D.**

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References

[1] L. Alessandrini and G. Bassanelli, *Positive $\partial\bar\partial$-closed currents and non-Kähler geometry*, J. Geom. Analysis, **2**(1992), 291-316.

[2] L. Alessandrini and G. Bassanelli, *Smooth proper modification of compact Kähler manifolds*, Proc. Intern. Workshop on Complex Analysis, Wupertal 1990, Complex Analysis, Aspects of Math., **17**(1991), 1-7.

[3] L. Alessandrini and G. Bassanelli, *Metric properties of manifolds bimeromorphic to compact Kähler manifolds*, J. Diff. Geometry, **37**(1993), 95-121.

[4] B. Alexandrov, G. Grantcharov and S. Ivanov, *Curvature properties of twistor spaces of quaternionic Kähler manifolds*, J. Geometry, **62**(1998), 1-12.

[5] V. Apostolov, G. Grantcharov and S. Ivanov, *Hermitian structures on twistor spaces*, Ann. Glob. Anal. Geom., **16**(1998), 291-308.

[6] H. Akbar-Zadeh, *Transformations holomorphiquement projectives des variétés hermitiennes et kählériennes*, J. Math. pures et appl., **67**(1988), 237-261.

[7] M.F. Atiyah, N.J. Hitchin, and I.M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London, Ser. A, **362**(1978), 425-461.

[8] A. Balas, *Compact Hermitian manifolds of constant holomorphic sectional curvature.*, Math. Z., **189**(1985), 193-210.

[9] A. Besse, *Einstein manifolds*, Springer, Berlin, 1987.

[10] J. Davidov and O. Muškarov, *On the Riemannian curvature of a twistor space*, Acta Math. Hung., **58**(1991), 319-332.

[11] J. Davidov, O. Muškarov and G. Grantcharov, *Curvature properties of twistor spaces*, Preprint.

[12] T. Friedrich and R. Grunevald, *On Einstein metrics on the twistor of a four-dimensional Riemannian manifold*, Math. Nachr., **106** (1985), 55-60.

[13] T. Friedrich and R. Kurke, *Compact four-dimensional self-dual Einstein manifolds with positive scalar curvature*, Math. Nachr., **123**(1982), 271-299.

[14] P. Gauduchon, *Fibrés hermitiennes à endomorphisme de Ricci non-négatif.*, Bull. Soc. Math. Fr., **105**(1977), 113-140.

[15] P. Gauduchon, *Structures de Weyl et théorèmes d'anulation sur une variété conforme autoduale*, Ann. Scuola Norm. Sup. Pisa, Serie IV, vol. XVIII, Fasc. 4(1991), 563-629.

[16] S. Goldberg, *Curvature and homology*, New-York, Academic Press, 1962.

[17] N.J. Hitchin, *Kählerian twistor spinors*, Proc. London Math. Soc., **43**(1981), 133-150.

[18] G. Jensen and M. Rigoli, *Twistor and Gauss lifts of surfaces in four manifolds*, Recent developments in Geometry, Contemporary Mathematics, **101**(1989), 197-232.

[19] S. Kobayashi, *On the automorphism group of a certain class of algebraic manifolds*, Tohoku Math. J., **11** (1959), 184-190.

[20] S. Kobayashi, *Transformation groups in Differential Geometry*, Springer-Verlag, Berlin, Heidelberg, New-York, 1972.

[21] S. Kobayashi, *Differential Geometry of complex vector bundles*, Iwanami Shoten, Publishers and Princeton Univ. Press., 1987.

[22] S. Kobayashi and H. Wu, *On holomorphic sections of certain Hermitian vector bundles*, Math. Ann., **189**(1970), 1-4.

[23] M.-L. Michelson, *On the existence of special metrics in complex geometry*, Acta Math., **143** (1983), 261-295.

[24] O. Muškarov, *Structures presque hermitiennes sur des espaces twistoriels et leurs types*, C. R. Acad. Sci. Paris, Sér I Math. **305**(1987), 307-309.

[25] T. Nitta and M. Takeuchi, *Contact structures on twistor spaces*, J. Math. Soc. Japan, **39**(1987), 139-162.

[26] M. Pontecorvo, *Complex structures on quaternionic manifolds*, Diff. Geom. Appl., **4**(1994), 163-177.

[27] S. Salamon, *Quaternionic Kähler manifolds*, Invent. Math., **67**(1982), 143-171.

[28] H. Wu, *Bochner technique in Differential Geometry*, Math. Reports, vol. **3**, part 2, 1988.

[29] A. Vitter, *Self-dual Einstein metrics*, Nonlinear Problems in Geometry, Contemporary Mathematics, **51**(1986), 113-120.

[30] K. Yano, *The theory of Lie derivatives and its applications*, Amsterdam, North-Holland, 1957.

[31] K. Yano, *Differential geometry on Complex and Almost Complex spaces*, Pergamon Press, 1965.

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