Dirac operators on Hermitian spin surfaces

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Abstract

We prove the conformal invariance of the dimension of the kernel of any of the self-adjoint Dirac operators associated to the canonical Hermitian connections on Hermitian spin surface. In the case of surface of non-negative conformal scalar curvature we estimate the first eigenvalue of the self-adjoint Dirac operator associated to the Chern connection and list the surfaces on which its kernel is non-trivial.

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1 Introduction

In [9] Hitchin has proved that the dimension of the kernel of the Dirac operator is conformal invariant. By the well-known vanishing theorem of Lichnerowicz this kernel is zero if the scalar curvature is non-negative and non-identically zero. When the scalar curvature is zero the harmonic spinors (i.e. the spinors in the kernel of the Dirac operator) are actually parallel spinors. The classification of the complete simply connected irreducible spin manifolds admitting parallel spinor is given by Hitchin [9] and Wang [17]. When the scalar curvature is positive an estimate for the first eigenvalue of the Dirac operator is found by Friedrich [5]. For Kähler manifolds this estimate is bettered by Kirchberg [11, 12]. In the 4-dimensional case (in which we are interested in this paper) the Kähler manifolds of smallest possible first eigenvalue are classified by Friedrich [6].

It is well-known that in the Kähler case the Dirac operator coincides with the Dolbeault operator of the square root of the canonical line bundle which determines the spin structure. Thus, it is natural to study the above questions in the case of the Dolbeault operator on Hermitian spin manifolds. For Hermitian spin surfaces this is done in [1]. Another way for generalization comes from the fact that on Kähler manifolds the Levi-Civita connection coincides with the Chern connection. Although generally the Dirac operator of the Chern connection is not self-adjoint on non-Kähler Hermitian manifolds (as shown in [13] it is self-adjoint iff the Lee form is zero), there is a natural self-adjoint operator associated to it. This operator is a particular case of the construction of Bismut [3] which associates to any metric connection with torsion T a self-adjoint Dirac operator D_T. In fact D_T is the sum of the Dirac operator of the connection with its adjoint, divided by 2. A Weitzenböck formula for D_T is also proved in [3].

Now we recall the definition of the set of canonical Hermitian connections, described by Gauduchon [8]. This set is the affine line of connections $\nabla^t, t \in R$, which is determined by the projection ∇^0 of the Levi-Civita connection on the affine space of Hermitian connections, and the Chern connection ∇^1.

Our first observation is that on Hermitian spin surfaces the dimension of the kernel of D_{T^t} is a conformal invariant (Proposition 3.1). Further, we prove the following:

Theorem 1.1 *Let (M,g,J) be a compact Hermitian spin surface of non-negative conformal scalar curvature and $\frac{1-\sqrt{3}}{2} < t < \frac{1+\sqrt{3}}{2}$. Then the kernel of D_{T^t} is non-zero iff (M,g,J) is conformally equivalent to a $K3$-surface or a flat torus with their hyper-Kähler metrics.*

Recall that the conformal scalar curvature of a Hermitian surface is the scalar curvature of the corresponding Weyl connection. In the following theorem we estimate the first eigenvalue of D_{T^1} on compact Hermitian spin surface of non-negative conformal scalar curvature:

Theorem 1.2 *Let (M,g,J) be a compact Hermitian spin surface of non-negative conformal scalar curvature. Then the first eigenvalue λ of D_{T^1} satisfies the inequality*

$$\lambda^2 \geq \frac{1}{6}\inf_M s, \qquad (1.1)$$

where s is the scalar curvature of g. The equality in (1.1) is attained iff (M,g,J) is one of the following manifolds:

(i) a $K3$-surface or a flat torus with their hyper-Kähler metrics

(ii) a Hopf surface with a metric of constant scalar curvature in the conformal class of the standard locally conformally flat metric and which is coordinate quaternionic with the opposite

orientation (cf. [4]) and whose fundamental group is $\mathbf{Z_m} \times \mathbf{Z}$ *(i.e. as a hypercomplex manifold with the opposite orientation it is one of the surfaces in the case (i) of Proposition 8 in [10]).*

Remark: By calculations similar to those in [1] we obtain that on compact Hermitian spin surfaces of positive conformal scalar curvature k any eigenvalue λ of D_{T^1} satisfies the inequality

$$\lambda^2 \geq \frac{1}{3}\mathrm{inf}_M k. \tag{1.2}$$

We do not know if there exist surfaces on which (1.2) turns into equality. Note that this is impossible on Kähler surfaces because of the estimate of Kirchberg [12].

2 Preliminaries

Let (M, g, J) be a Hermitian surface with complex structure J and compatible metric g. Denote by Ω the Kähler form, defined by $\Omega(X, Y) = g(X, JY)$. The volume form of g is $\omega = \frac{1}{2}\Omega \wedge \Omega$. It is well-known that $d\Omega = \theta \wedge \Omega$, where $\theta = \delta\Omega \circ J$ is the Lee form of (M, g, J). Recall that (M, g, J) is Kähler iff $\theta = 0$; locally conformally Kähler iff $d\theta = 0$; globally conformally Kähler iff $\theta = df$ for a smooth function f on M (in this case $e^{-f}g$ is a Kähler metric). Let ∇ be the Levi-Civita connection of g and R and s – its curvature tensor and scalar curvature respectively (for the curvature tensor we adopt the following definition: $R(X, Y, Z, W) = g([\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, W))$. Recall that the *-Ricci tensor ρ^* and the *-scalar curvature s^* of M are defined by

$$\rho^*(X, Y) = \sum_{i=1}^{4} R(e_i, X, JY, Je_i) = -\frac{1}{2}\sum_{i=1}^{4} R(X, JY, e_i, Je_i)$$

$$s^* = \sum_{i=1}^{4} \rho^*(e_i, e_i),$$

where here and in the following $\{e_i\}$ is a local orthonormal frame of the tangent bundle TM and $\{e^i\}$ is its dual frame.

We also have (cf. [16])

$$s - s^* = 2\delta\theta + |\theta|^2. \tag{2.3}$$

Note that on Kähler manifolds the Ricci and *-Ricci tensors coincide; in particular $s = s^*$.

Now consider the Weyl connection determined by the Hermitian structure on M, i.e. the unique torsion-free connection ∇^W such that $\nabla^W g = \theta \otimes g$. The conformal scalar curvature k is defined to be the scalar curvature of ∇^W. Equivalently, k is the *-scalar curvature of the self-dual Weyl tensor W_+, multiplied by $\frac{3}{2}$. The Weyl connection is invariant under conformal changes of the metric since if $\widetilde{g} = e^f g$ then $\widetilde{\theta} = \theta + df$. Hence,

$$\widetilde{k} = e^{-f}k. \tag{2.4}$$

We also have

$$k = \frac{3s^* - s}{2}.$$

Thus, it follows from (2.3) that

$$k = s - 3\delta\theta - \frac{3}{2}|\theta|^2. \tag{2.5}$$

Recall the definition of the set of canonical Hermitian connections [8]: For a real number t the connection ∇^t is defined by

$$g(\nabla^t_X Y, Z) = g(\nabla_X Y, Z) - \frac{1}{2} g((\nabla_{JX} J)Y, Z) + \frac{t}{4}(d\Omega(JX, JY, JZ) + d\Omega(JX, Y, Z))$$

or equivalently

$$\begin{aligned} \nabla^t_X Y &= \nabla_X Y - \frac{t+1}{4}\theta(Y)X + \frac{t-1}{4}\theta(JY)JX - \frac{t}{2}\theta(JX)JY \\ &+ \frac{t+1}{4} g(X,Y)\theta^\# - \frac{t-1}{4} g(X, JY)J\theta^\#, \end{aligned} \tag{2.6}$$

where $\theta^\#$ is the vector field dual to θ. We shal denote by $J\theta$ the form dual to $J\theta^\#$, i.e. $J\theta = -\theta \circ J$.

The canonical Hermitian connections form an affine line (degenerating to a point in the Kähler case) determined by ∇^0 - the projection of the Levi-Civita connection into the affine space of all Hermitian connections, and ∇^1, which coincides with the Chern connection.

From now on we assume that (M, g, J) is spin manifold. Denote by ΣM its spinor bundle and let $\mu : T^*M \otimes \Sigma M \longrightarrow \Sigma M$ be the Clifford multiplication. Identifying T^*M and TM via the metric we shall also consider μ as a map from $TM \otimes \Sigma M$ into ΣM. We shall often write the Clifford multiplication by juxtaposition, i.e.

$$\mu(\alpha \otimes \psi) = \alpha\psi, \qquad \mu(X \otimes \psi) = X\psi, \qquad \alpha \in T^*M, \quad X \in TM, \quad \psi \in \Sigma M.$$

Since (M, g, J) is a Hermitian manifold, the choice of a spin structure on it is equivalent to the choice of a square-root $K^{\frac{1}{2}}$ of the canonical line bundle K [9]. Thus for the corresponding spinor bundle we have $\Sigma M = \Lambda^{0,\bullet}M \otimes K^{\frac{1}{2}}$, where $\Lambda^{0,\bullet}M = \Lambda^{0,0}M \oplus \Lambda^{0,1}M \oplus \Lambda^{0,2}M$. In particular, the spinor bundle ΣM splits as follows:

$$\Sigma M = \Sigma_0 M \oplus \Sigma_1 M \oplus \Sigma_2 M, \tag{2.7}$$

where $\Sigma_r M = \Lambda^{0,r}M \otimes K^{\frac{1}{2}}$ is the eigensubbundle with respect to the eigenvalue $(2-2r)i$ of the Clifford action of the Kähler form Ω on ΣM (cf. [11]).

The half-spinor bundles $\Sigma_\pm M$ are the eigensubbundles of the volume form ω with respect to its eigevalues ∓ 1 and we have

$$\Sigma_+ M = \Sigma_0 M \oplus \Sigma_2 M, \qquad \Sigma_- M = \Sigma_1 M.$$

Let $p_r : \Sigma M \longrightarrow \Sigma_r M$, $r = 0, 1, 2$, be the projections with respect to the splitting (2.7). For convenience we denote $\Sigma_{-1}M = \Sigma_3 M = 0$ and $p_{-1} = p_3 = 0$. Recall [11] that for $X \in TM$ $X\Omega - \Omega X = 2JX$ as endomorphisms of ΣM and for $\psi \in \Sigma_r M$

$$X\psi = p_{r-1}X\psi + p_{r+1}X\psi, \qquad JX\psi = -ip_{r-1}X\psi + ip_{r+1}X\psi, \tag{2.8}$$

$$p(X)\psi = p_{r+1}X\psi, \qquad \bar{p}(X)\psi = p_{r-1}X\psi, \tag{2.9}$$

where $p(X) = \frac{1}{2}(X - iJX)$, $\bar{p}(X) = \frac{1}{2}(X + iJX)$.

Any metric connection in the tangent bundle gives rise to a metric connection in the spinor bundle and it is easy consequence of (2.6) that

$$\nabla^t_X \psi = \nabla_X \psi + \frac{1}{8}((t+1)X\theta + (t-1)JXJ\theta + 2t\theta(X) + 2t\theta(JX)\Omega)\psi \tag{2.10}$$

for $X \in \Gamma(TM)$, $\psi \in \Gamma(\Sigma M)$. In particular, we have

$$\nabla^t_X \psi_1 = \nabla_X \psi_1 + \frac{t}{4}(X\theta + \theta(X))\psi_1, \qquad X \in \Gamma(TM), \quad \psi_1 \in \Gamma(\Sigma_1 M). \tag{2.11}$$

This follows from (2.10) and

$$JXJY\psi_1 = XY\psi_1, \qquad X, Y \in \Gamma(TM), \quad \psi_1 \in \Gamma(\Sigma_1 M). \tag{2.12}$$

The Kähler form Ω is parallel with respect to any Hermitian connection, so the Hermitian connections preserve the splitting (2.7).

Let ∇' be any metric connection on (M, g) with torsion T'. The Dirac operator $D' : \Gamma(\Sigma M) \longrightarrow \Gamma(\Sigma M)$ of ∇' is defined by $D' = \mu \circ \nabla'$ or equivalently by $D'\psi = \sum_{i=1}^4 e^i \nabla'_{e_i}\psi$. Let $D_{T'} = \frac{1}{2}(D' + D'^*)$. Then $D_{T'}$ is self-adjoint and the following formulae are proved in [3], Theorem 1.10:

$$D_{T'} = D + \frac{1}{4}AT', \tag{2.13}$$

$$D^2_{T'} = (\overline{\nabla}')^*\overline{\nabla}' + \frac{s}{4} + \frac{1}{4}dAT' - \frac{1}{8}|AT'|^2, \tag{2.14}$$

where D is the Dirac operator of the Levi-Civita connection, AT' is the antisymmetrization of T', $AT'(X, Y, Z) = g(T'(X, Y), Z) + g(T'(Y, Z), X) + g(T'(Z, X), Y)$, and the connection $\overline{\nabla}'$ is defined by

$$\overline{\nabla}'_X \psi = \nabla_X \psi + \frac{1}{4}(\iota_X AT')\psi, \qquad X \in \Gamma(TM), \quad \psi \in \Gamma(\Sigma M), \tag{2.15}$$

where ι_X is the interior multiplication operator. As shown in [8], the Dirac operator D^t of ∇^t is

$$D^t = D - \frac{3t}{4}\theta - \frac{2t-1}{4}J\theta \circ \Omega.$$

Hence,

$$D_{T^t} = D - \frac{2t-1}{4}(\theta + J\theta \circ \Omega). \tag{2.16}$$

We also have $AT^t = -(2t-1)J\theta \wedge \Omega$ (cf. [8]) and thus by (2.14) and (2.15) we obtain

$$D^2_{T^t} = (\overline{\nabla}^t)^*\overline{\nabla}^t + \frac{s}{4} - \frac{2t-1}{4}\delta\theta.\omega - \frac{(2t-1)^2}{8}|\theta|^2, \tag{2.17}$$

$$\overline{\nabla}^t_X \psi = \nabla_X \psi + \frac{2t-1}{4}(JXJ\theta + \theta(X) + \theta(JX)\Omega)\psi \tag{2.18}$$

for $X \in \Gamma(TM)$, $\psi \in \Gamma(\Sigma M)$.
Using (2.5) to exclude $\delta\theta$, we see from (2.17) that

$$D^2_{T^t}\psi_+ = (\overline{\nabla}^t)^*\overline{\nabla}^t\psi_+ + \left(\frac{t+1}{6}s - \frac{2t-1}{12}k - \frac{(2t-1)t}{4}|\theta|^2\right)\psi_+, \tag{2.19}$$

$$D^2_{T^t}\psi_- = (\overline{\nabla}^t)^*\overline{\nabla}^t\psi_- + \left(\frac{2-t}{6}s + \frac{2t-1}{12}k + \frac{(2t-1)(1-t)}{4}|\theta|^2\right)\psi_- \tag{2.20}$$

for $\psi_+ \in \Gamma(\Sigma_+ M)$, $\psi_- \in \Gamma(\Sigma_- M)$.

Recall (cf. [2]) that under a conformal change of the metric $\widetilde{g} = e^f g$ of a spin manifold (M, g) there is an identification $\widetilde{\ }$ of the spinor bundle ΣM of (M, g) and the spinor bundle $\widetilde{\Sigma}M$ of $(M, \widetilde{g})$ such that

$$\widetilde{X\psi} = \tilde{X}\widetilde{\psi}, \tag{2.21}$$

$$\widetilde{\alpha\psi} = \tilde{\alpha}\tilde{\psi}, \tag{2.22}$$

$$\tilde{\nabla}_{\tilde{X}}\tilde{\psi} = \widetilde{\nabla_X\psi} - \frac{1}{4}X df\tilde{\psi} - \frac{1}{4}Xf.\tilde{\psi}, \tag{2.23}$$

where $\psi \in \Gamma(\Sigma M)$, $\tilde{\psi} \in \Gamma(\tilde{\Sigma}M)$ is the spinor corresponding to ψ, $X \in TM$, $\tilde{X} = e^{-\frac{f}{2}}X$, $\alpha \in T^*M$, $\tilde{\alpha} = e^{\frac{f}{2}}\alpha$, and ∇ and $\tilde{\nabla}$ are the Levi-Civita connections of g and $\tilde{g}$ respectively.

Now, if (M,g,J) is a Hermitian surface, the Kähler form of $(M,\tilde{g},J)$ is $\tilde{\Omega} = e^f\Omega$ and it follows from (2.22) that

$$\widetilde{\Omega\psi} = \tilde{\Omega}\tilde{\psi}. \tag{2.24}$$

Thus the eigensubbundles $\Sigma_r M$ of Ω correspond to the eigensubbundles $\tilde{\Sigma}_r M$ of $\tilde{\Omega}$. The Lee form of $\tilde{g}$ is $\theta + df$ and hence by (2.16), (2.21)-(2.24) we obtain

$$\tilde{D}\tilde{\psi} = e^{-\frac{f}{2}}\widetilde{D\psi} + \frac{3}{4}df\tilde{\psi}, \tag{2.25}$$

$$\tilde{D}_{\tilde{T}^t}\tilde{\psi} = e^{-\frac{f}{2}}\widetilde{D_{T^t}\psi} - \frac{t-2}{2}df\tilde{\psi} - \frac{2t-1}{4}Jdf\widetilde{\Omega\psi}. \tag{2.26}$$

In the following we shall denote by $<\cdot,\cdot>$ and $|\cdot|$ pointwise inner products and norms and by $(\cdot,\cdot)$ and $\|\cdot\|$ – the global ones respectively.

3 Proof of Theorem 1.1 and Theorem 1.2

Throughout this section (M,g,J) is a compact Hermitian spin surface.

Proposition 3.1 *The dimension of the kernel of D_{T^t} is a conformal invariant.*

Proof: Since $D_{T^t}(\Gamma(\Sigma_\pm M)) \subset \Gamma(\Sigma_\mp M)$, it is enough to show that $dim\,Ker\,D_{T^t}|_{\Sigma_\pm M}$ is a conformal invariant. It follows from (2.26) and (2.8) that

$$\tilde{D}_{\tilde{T}^t}\widetilde{\psi_+} = e^{-\frac{f}{2}}\widetilde{D_{T^t}\psi_+} + \frac{t+1}{2}df\widetilde{\psi_+}, \qquad \psi_+ \in \Gamma(\Sigma_+M),$$

$$\tilde{D}_{\tilde{T}^t}\widetilde{\psi_-} = e^{-\frac{f}{2}}\widetilde{D_{T^t}\psi_-} - \frac{t-2}{2}df\widetilde{\psi_-}, \qquad \psi_- \in \Gamma(\Sigma_-M).$$

Hence,

$$\tilde{D}_{\tilde{T}^t}(e^{-\frac{t+1}{2}f}\widetilde{\psi_+}) = e^{-\frac{t+2}{2}f}\widetilde{D_{T^t}\psi_+}, \qquad \psi_+ \in \Gamma(\Sigma_+M),$$

$$\tilde{D}_{\tilde{T}^t}(e^{\frac{t-2}{2}f}\widetilde{\psi_-}) = e^{\frac{t-3}{2}f}\widetilde{D_{T^t}\psi_-}, \qquad \psi_- \in \Gamma(\Sigma_-M),$$

and these equalities prove Proposition 3.1. □

Proof of Theorem 1.1: By theorem of Gauduchon [7] there always exists a metric in the conformal class of g whose Lee form is co-closed. It follows from (2.4) and Proposition 3.1 that we can assume that g coincides with this metric. Hence, by (2.17) and (2.5) we obtain

$$\|D_{T^t}\psi\|^2 = \|\overline{\nabla}^t\psi\|^2 + \frac{1}{4}((k + \frac{3-(2t-1)^2}{2}|\theta|^2)\psi,\psi).$$

But $k + \frac{3-(2t-1)^2}{2}|\theta|^2 \geq 0$ for $\frac{1-\sqrt{3}}{2} < t < \frac{1+\sqrt{3}}{2}$. Thus it follows that $0 \neq \psi \in Ker\,D_{T^t}$ iff $\overline{\nabla}^t\psi = 0$, $k = 0$ and $\theta = 0$. The condition $\theta = 0$ means that (M,g,J) is Kähler and in this case

$\overline{\nabla}^t = \nabla$. The existence of a parallel spinor with respect to the Levi-Civita connection implies that the manifold is a $K3$-surface or a flat torus with their hyper-Kähler metrics (cf. [9] or [14]). Conversely, any of these manifolds admits a parallel spinor in the spin structure given by the trivial square-root of the canonical line bundle. $\square$

We begin the proof of Theorem 1.2 by the following simple lemma:

Lemma 3.2 *Let $\lambda \neq 0$ be an eigenvalue of D_{T^t}. Then there exist eigenspinors $\psi_+ \in \Gamma(\Sigma_+ M)$ and $\psi_- \in \Gamma(\Sigma_- M)$ for the eigenvalue λ^2 of $D^2_{T^t}$.*

Proof: Let $\psi \neq 0$ be an eigenspinor of D_{T^t} for λ and $\psi = \psi_+ + \psi_-$, $\psi_\pm \in \Gamma(\Sigma_\pm M)$. Then $D_{T^t}\psi_\pm = \lambda\psi_\mp$. Thus $\psi_\pm \neq 0$ and $D^2_{T^t}\psi_\pm = \lambda^2\psi_\pm$. $\square$

Proof of Theorem 1.2: When $\inf_M s \leq 0$ the inequality (1.1) is trivially satisfied. So we can assume that $\inf_M s > 0$. Since neither of the manifolds in Theorem 1.1 admits a metric of non-negative non-identically zero scalar curvature (the $K3$-surfaces has non-zero $\hat{A}$-genus and for the torus see [15]), we obtain that $Ker\, D_{T^1} = 0$. Hence $\lambda \neq 0$ and by Lemma 3.2 we have an eigenspinor $\psi_- \in \Gamma(\Sigma_- M)$ of $D^2_{T^1}$ with respect to λ^2. Thus (2.20) yields

$$\lambda^2\|\psi_-\|^2 = (D^2_{T^1}\psi_-, \psi_-) = \|\overline{\nabla}^1\psi_-\|^2 + ((\frac{s}{6} + \frac{k}{12})\psi_-, \psi_-)$$

and since $k \geq 0$, we obtain that $\lambda^2 \geq \frac{1}{6}\inf_M s$.

Now we proceed with the proof of the second part of the theorem. When $\inf_M s = 0$ the kernel of D_{T^1} is non-trivial. Then it follows from Theorem 1.1 that (M, g, J) is conformally equivalent to a $K3$-surface or a flat torus. Since the scalar curvature is non-negative, we obtain that (M, g, J) is one of the manifolds in (i).

When $s > 0$ the calculations in the first part of the proof show that there is an equality in (1.1) iff $s = const$, $k = 0$ and $\overline{\nabla}^1\psi_- = 0$. It follows from (2.11), (2.12), (2.18) that on $\Sigma_- M$ $\overline{\nabla}^1 = \nabla^1$ and hence the condition $\overline{\nabla}^1\psi_- = 0$ may be replaced by $\nabla^1\psi_- = 0$. Note that the Chern connection restricted to sections of $\Sigma_- M$ coincides with the connection considered in [14] and called there "the Weyl connection". Hence, as proved in [14], a parallel spinor in $\Sigma_- M$ with respect to the Chern connection gives rise to a hyper-Hermitian structure on M with the opposite orientation and Lee form θ. A hyper-Hermitian 4-manifold is conformally equivalent to a $K3$-surface or a flat torus or a coordinate quaternionic Hopf surface [4]. But $s > 0$ and thus (M, g, J) with the opposite orientation has to be a coordinate quaternionic Hopf surface, whose Lee form is the same as that of J. Now it is easy to see that the fundamental group of M has to be $\mathbf{Z_m} \times \mathbf{Z}$. This shows that (M, g, J) is one of the Hopf surfaces in item (i) of Proposition 8 in [10]. Conversely, any of these surfaces admits a parallel spinor in $\Sigma_- M$ with respect to the Chern connection in the spin structure given by the trivial square-root of the canonical line bundle. $\square$

Remark 1: Let (M, g, J) be as in Theorem 1.2. If $0 < t < \frac{1}{2}$, then it follows from (2.19) that the first eigenvalue λ of D_{T^t} satisfies the inequality

$$\lambda^2 \geq \frac{t+1}{6}\inf_M s.$$

The equality is attained iff $s = const$, $k = 0$, $\theta = 0$ and there exists $\psi_+ \in \Gamma(\Sigma_+ M)$ such that $\overline{\nabla}^t\psi_+ = 0$. But the condition $\theta = 0$ means that (M, g, J) is Kähler and in this case $\overline{\nabla}^t = \nabla$, so as

in the proof of Theorem 1.1 the manifold is a $K3$-surface or a flat torus with their hyper-Kähler metrics.
Similarly, for $\frac{1}{2} < t < 1$ it follows from (2.20) that the first eigenvalue λ of D_{T^t} satisfies

$$\lambda^2 \geq \frac{2-t}{6}\mathrm{inf}_M s$$

and the equality is attained iff the manifold is hyper-Kähler.
The above estimates can be extended for $t = \frac{1}{2}$ (and the condition $k \geq 0$ is not necessary), but in this case (2.16) and (2.18) show that $D_{T^{\frac{1}{2}}} = D$, $\overline{\nabla}^{\frac{1}{2}} = \nabla$ and (2.17) is the well-known Lichnerowicz formula for D. So, again the equality occurs only on hyper-Kähler manifolds.
Now let $t = 0$. Comparing (2.16) with formula (2.16) in [1], we see that D_{T^0} coincides with the Dolbeault operator of $K^{\frac{1}{2}}$. Thus from Theorem 1.2 in [1] we obtain that the first eigenvalue of D_{T^0} also satisfies (1.1) and the equality is attained iff (M, g, J) is a $K3$-surface or a flat torus with their hyper-Kähler metrics, or a Hopf surface with a metric of constant scalar curvature in the conformal class of the standard locally conformally flat metric and which is coordinate quaternionic with the same orientation.

Remark 2: Let $\pi_r : T^*M \otimes \Sigma_r M \longrightarrow Ker\, \mu|_{T^*M\otimes\Sigma_r M}$, $r = 0, 1, 2$, be the orthogonal projections. It is easy to see that

$$\pi_r(\alpha \otimes \psi) = \alpha \otimes \psi + \frac{1}{2(r+1)} \sum_{i=1}^{4} e^i \otimes \bar{p}(e^i) p(\alpha)\psi + \frac{1}{4-2(r-1)} \sum_{i=1}^{4} e^i \otimes p(e^i) \bar{p}(\alpha)\psi, \tag{3.27}$$

where $p(\alpha) = \frac{1}{2}(\alpha - iJ\alpha)$, $\bar{p}(\alpha) = \frac{1}{2}(\alpha + iJ\alpha)$.
The twistor operators of the Hermitian connection ∇^t are the differential operators $P_r^t : \Gamma(\Sigma_r M) \longrightarrow \Gamma(T^*M \otimes \Sigma_r M)$ defined by $P_r^t = \pi_r \circ \nabla^t$. By analogy with the Kähler twistor spinors of [12] we shall call the spinors in the kernel of P_r^t *Hermitian twistor spinors with respect to* ∇^t. It follows from (3.27) that

$$P_1^t \psi_1 = \nabla^t \psi_1 + \frac{1}{4} \sum_{i=1}^{4} e^i \otimes e_i D^t \psi_1, \qquad \psi_1 \in \Gamma(\Sigma_1 M)$$

By calculations similar to those in [1] we obtain

$$\|D_{T^1}\psi_1\|^2 = \frac{4}{3}\|P_1^3 \psi_1\|^2 + \frac{1}{3}(k\psi_1, \psi_1).$$

Thus if $k > 0$ any eigenvalue λ of D_{T^1} satisfies the inequality (1.2) and the equality is attained iff $k = const$ and there exists Hermitian twistor spinor $\psi_1 \in \Gamma(\Sigma_1 M)$ with respect to ∇^3. We do not know if there exist surfaces on which (1.2) turns into equality. Note that this is impossible on Kähler surfaces because of the estimate of Kirchberg [12], according to which the first eigenvalue λ of D satisfies $\lambda^2 \geq \frac{1}{2}\mathrm{inf}_M s$.
For $t = 0$ it follows from Theorem 1.3 in [1] that if $k > 0$, then any eigenvalue λ of D_{T^0} satisfies

$$\lambda^2 \geq \frac{1}{2}\mathrm{inf}_M k.$$

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