Riemannian manifolds in which certain curvature operator has constant eigenvalues along each circle

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Abstract

Riemannian manifolds for which a natural curvature operator has constant eigenvalues on circles are studied. A local classification in dimensions two and three is given. In the three dimensional case one gets all locally symmetric spaces and all Riemannian manifolds with the constant principal Ricci curvatures $r_1 = r_2 = 0, r_3 \neq 0$, which are not locally homogeneous, in general.

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1 Introduction

Curvature is a fundamental notion of differential geometry. A useful technique to describe the curvature along a geodesic in a Riemannian manifold is the use of the Jacobi operator. In [1] J.Berndt and L.Vanhecke have introduced the so called C-*spaces* and P-*spaces* as a generalizations of locally symmetric spaces. A C-space (resp. P-space) is by definition a Riemannian manifold for which the Jacobi operator has constant eigenvalues (resp. parallel eigenspaces) along every geodesic.

In the present note we describe the curvature along a unit circle using the curvature operator. A unit circle in a Riemannian manifold is a smooth curve with first curvature equal to one and all other curvatures equal to zero (Cf. K.Nomizu and K.Yano [8]). We consider Riemannian manifold for which natural curvature operators defined along every unit circle have constant eigenvalues along the circle. Such a Riemannian manifold is called an O-*space*. For example, every locally symmetric space is an O-space.

An equivalent point of view is the following. Let $c(t)$ be a unit circle on a Riemannian manifold $(\mathbf{M}, g)$ and let $T_{c(t)}\mathbf{M}$ denote the tangent space at every point $c(t)$. We may consider the "extrinsic circle flow" on the orthonormal 2-frame bundle of $\mathbf{M}$ whose orbits are defined by $t \longrightarrow (\dot{c}(t), n(t)) \in T_{c(t)}\mathbf{M} \times T_{c(t)}\mathbf{M}$, where $\dot{c}(t)$ and $n(t)$ denote the tangent vector field and the first normal field of a unit circle $c(t)$, respectively. An O- space is a space such that the skew-symmetric curvature operators $R(\dot{c}(t), n(t))$ on $T_{c(t)}\mathbf{M}$ have constant eigenvalues along every orbit.

The main result of this note is the following

Theorem 1.1 *Let* $(\mathbf{M},g)$ *be a 3-dimensional O-space of class* C^∞. *Then* $\mathbf{M}$ *is locally isometric almost everywhere (that is on an open and dense subset of* $\mathbf{M}$*) to one of the following spaces:*

i) a space of constant Riemannian sectional curvature;

ii) a Riemannian product of the form $\mathbf{M}_2 \times \mathbf{R}$, *where* $\mathbf{M}_2$ *is a 2-dimensional Riemannian manifold of constant sectional curvature.*

iii) a Riemannian manifold with constant principal Ricci curvatures $r_1 = r_2 = 0, r_3 \neq 0$.

Conversely, any 3-dimensional Riemannian manifold of type i), ii) or iii) is an O-space.

J. Milnor has studied in [7] 3-dimensional Lie groups with left invariant Riemannian metrics with constant principal Ricci curvatures $r_1 = r_2 = 0, r_3 \neq 0$. He has found a lot of examples of such spaces which are locally homogeneous but not locally symmetric ($SU(2)$ with its special left invariant metric for example). By Theorem 1.1 these spaces are examples of O- spaces which are locally homogeneous but not locally symmetric.

According to Singer [10] a Riemannian manifold $(\mathbf{M}, g)$ is called *curvature homogeneous* if for any pair of points $p, q \in \mathbf{M}$ there is a linear isometry $F : T_p\mathbf{M} \longrightarrow T_q\mathbf{M}$ between the corresponding tangent spaces such that $F^*R_q = R_p$ (where R denotes the curvature tensor of type $(0,4)$). It is well known that a 3- dimensional Riemannian space is curvature homogeneous iff it has constant principal Ricci eigenvalues. Recently, O.Kowalski has constructed in [5] an explicit example of Riemannian 3-manifold with constant principal Ricci curvatures $r_1 = r_2 = 0, r_3 \neq 0$ which is not locally homogeneous. By Theorem 1.1 this is an example of O-space which is curvature homogeneous but not locally homogeneous. We describe a special kind of it.

Example 1. [5]. We consider $\mathbf{R}^3$ with the following non-homogeneous Riemannian metric

$$ds^2 = e^{-2y}dz^2 + \Big[(e^y dx + z(e^{-y} - x^2 e^y)dz\Big]^2 + (dy + 2xzdz)^2,$$

where x, y, z are the standard coordinates in $\mathbf{R}^3$. Its principal Ricci curvatures are given by: $r_1 = r_2 = 0, r_3 = -2$.

By Theorem 4 in [1] this O-space is not a C-space. We show (see Remark 2 at the end of this note) that this example is not also a P-space. Moreover, Theorem 1.1 and Theorem 7 in [1] imply (see also Remark 2) that a 3-dimensional Riemannian manifold $\mathbf{M}$ is locally symmetric iff $\mathbf{M}$ is an O-space and $\mathbf{M}$ is a P- space simultaneously. On the other hand, a 3-dimensional Riemannian manifold $\mathbf{M}$ which is both O- and P-space may not be a locally symmetric space. (For example, $SU(2)$ with its special left invariant metric is an O-space and it is also a C-space by Theorem 4 in [1]).

O-spaces can be regarded as a non-trivial generalization of locally symmetric spaces since every locally symmetric space is an O-space but, in dimension 3, there is a family of $\mathcal{O}$-spaces which are not even locally homogeneous and depend essentially on two arbitrary functions of one variable (see [5]). Theorem 1.1 implies that every 3-dimensional O-space is curvature homogeneous but there is a lot of curvature homogeneous (and even locally homogeneous) 3- manifolds which are not O-spaces. See [5, 11, 6].

2 Characterizations of O-spaces

Let $(\mathbf{M}, g)$ be an n-dimensional Riemannian manifold and ∇ the Levi-Civita connection of the metric g. We denote by $T_p\mathbf{M}$ the tangential space at a point p of $\mathbf{M}$. The curvature operator is defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for every smooth vector fields X, Y, Z, V on $\mathbf{M}$. The curvature tensor of type (0,4) is defined by $R(X,Y,Z,V) := g(R(X,Y)Z,V)$.

A smooth curve $c(t) : I_\epsilon \to \mathbf{M}, I_\epsilon = (-\epsilon, \epsilon), \epsilon > 0$, with tangent vector field $\dot{c}$, parameterized by the arc length, is said to be a circle of curvature k_1 if its first curvature k_1 is a constant different from zero and all other curvatures are zeros (see [8]). A unit circle is a circle with curvature $k_1 = 1$. For a unit circle we have

$$\nabla_{\dot{c}}\dot{c} = n; \qquad \nabla_{\dot{c}}n = -\dot{c}. \tag{2.1}$$

The unit vector field n is the first normal of $c(t)$. All other normals orthogonal to $\dot{c}$ and n are parallel along $c(t)$. The differential equations for a unit circle is equivalent to

$$\nabla_{\dot{c}}\nabla_{\dot{c}}\dot{c} + \dot{c} = 0. \tag{2.2}$$

If dim $\mathbf{M} \geq 2$ then for every point $p \in \mathbf{M}$ and every two orthonormal vectors $u, v \in T_p\mathbf{M}$ there exists locally a unique unit circle $c(t)$, parameterized by the arc length and satisfying the following initial conditions

$$c(0) = p, \qquad \dot{c}(0) = u, \qquad (\nabla_{\dot{c}}\dot{c})(0) = v. \tag{2.3}$$

Let $c(t)$ be a unit circle on $\mathbf{M}$. We consider the families of smooth skew-symmetric linear operators along $c(t)$ defined by

$$\kappa_c(X) := R(\dot{c}, \nabla_{\dot{c}}\dot{c})X, \qquad \kappa'_c(X) := (\nabla_{\dot{c}}R)(\dot{c}, \nabla_{\dot{c}}\dot{c})X \tag{2.4}$$

for every vector field X along $c(t)$.

It follows from (2.1) and (2.4) that

$$\kappa_c'(X) = \nabla_{\dot{c}}\kappa_c(X) - \kappa_c(\nabla_{\dot{c}}X). \tag{2.5}$$

We have the following characterization of locally symmetric spaces:

Proposition 2.1 *For an n-dimensional ($n \geq 2$) Riemannian manifold ($\mathbf{M}$,g) the following conditions are equivalent:*

i) ($\mathbf{M}$,g) is a Riemannian locally symmetric space;

ii) For every unit circle c on $\mathbf{M}$ we have $\kappa_c' = 0$;

iii) For every unit circle c on $\mathbf{M}$ we have $\kappa_c \circ \nabla_{\dot{c}} = \nabla_{\dot{c}} \circ \kappa_c$;

iv) For every unit circle c on $\mathbf{M}$ the operator κ_c transforms a vector field parallel along c to a vector field parallel along c;

v) For every unit circle c on $\mathbf{M}$ the matrix of κ_c with respect to a basis of vector fields parallel along c is invariant i.e. the matrix has constant entries.

Proof. The implication i) $\rightarrow$ ii) is clear.

To prove the converse, let u, v be two orthonormal vectors at a point $p \in \mathbf{M}$. Let $c(t)$ be the unit circle on $\mathbf{M}$ determined by the initial conditions (2.3). For every $w \in T_p\mathbf{M}$, we get from $\kappa_c' = 0$ the equality

$$(\nabla_u R)(u, v)w = 0. \tag{2.6}$$

By the well known properties of the Riemannian curvature operator we can conclude that (2.6) is valid for arbitrary $u, v, w \in T_p\mathbf{M}$. If $z \in T_p\mathbf{M}$, replacing v by $v + z$ in (2.6) and using the second Bianchi identity we get $\nabla R = 0$ which proves i).

The equivalence of ii), iii), iv) and v) follows from (2.5). **Q.E.D.**

We recall some of the basic properties of the skew-symmetric linear operators.

Let A be a skew-symmetric linear operator on a Euclidean vector space E. If A has a real eigenvalue then it has to be zero. If A possesses a non-zero eigenvalue of multiplicity m then it has to be of the form $\sqrt{-1}a, a \in \mathbf{R}$. Then $-\sqrt{-1}a$ is also an eigenvalue of A of the same multiplicity. Let A has eigenvalues $0, \sqrt{-1}a_1, -\sqrt{-1}a_1, ..., \sqrt{-1}a_l, -\sqrt{-1}a_l$ of multiplicities $m, m_1, m_1, ..., m_l, m_l$, respectively. Then there exists an orthonormal basis $Z_1, ..., Z_m, X_1^{(1)}, Y_1^{(1)}, ..., X_{m_1}^{(1)}, Y_{m_1}^{(1)}, ..., X_1^{(l)}, Y_1^{(l)}, X_{m_l}^{(l)}, Y_{m_l}^{(l)}$ of E such that

$$A(Z_p) = 0, \qquad p = 1, ..., m, \tag{2.7}$$

$$A(X_s^{(j)}) = a_j Y_s^{(j)}, A(Y_s^{(j)}) = -a_j X_s^{(j)}, \quad j = 1, ..., l; s = 1, ...m_j.$$

Such a basis of E is known as a Jordanian basis for A.

Definition 2.1 A Riemannian manifold ($\mathbf{M}$,g) is said to be *an O-space* if for any unit circle $c(t)$ the operator κ_c has constant eigenvalues along $c(t)$.

Definition 2.2 A Riemannian manifold ($\mathbf{M}$,g) is said to be a *T-space* if for any unit circle $c(t)$ the operator κ_c has Jordanian basis parallel along $c(t)$ (at least locally around almost every point of $c(t)$).

Remark 1. It follows from Proposition 2.1 that a smooth Riemannian manifold $\mathbf{M}$ is a locally symmetric Riemannian manifold iff $\mathbf{M}$ is an O-space and $\mathbf{M}$ is a T-space simultaneously.

Due to this observation it is natural to study the O-spaces and the T-spaces separately. This leads to two natural generalizations of locally symmetric Riemannian manifolds.

In this paper we consider the O-spaces. The T-spaces we study in [4].

Our further considerations are based on the following simple observation: If a skew-symmetric linear operator A on an Euclidean vector space E has eigenvalues $0, \sqrt{-1}a_1, -\sqrt{-1}a_1, ..., \sqrt{-1}a_l, -\sqrt{-1}a_l$ then the operator A^2 is self-adjoint linear operator and it has eigenvalues $0, -a_1^2, ..., -a_l^2$.

Proposition 2.2 *Let $(\mathbf{M},g)$ be an n-dimensional $(n \geq 2)$ Riemannian manifold. Then $\mathbf{M}$ is an O-space iff the self-adjoint operator κ_c^2 has constant eigenvalues along every unit circle c on $\mathbf{M}$.*

Further, for each point $p \in \mathbf{M}$ and for any two orthonormal tangent vectors u, v at p we define two skew-symmetric endomorphisms $\kappa_{u,v}, \kappa'_{u,v}$ of $T_p\mathbf{M}$ by

$$\kappa_{u,v} = R(u,v), \qquad \kappa'_{u,v} = (\nabla_u R)(u,v).$$

Then the endomorphisms $\kappa_{u,v}^2 = \kappa_{u,v} \circ \kappa_{u,v}$ and $(\kappa_{u,v}^2)' = \kappa'_{u,v} \circ \kappa_{u,v} + \kappa_{u,v} \circ \kappa'_{u,v}$ are self-adjoint endomorphisms of $T_p\mathbf{M}$.

We have

Proposition 2.3 *A Riemannian manifold $(\mathbf{M},g)$ $(dim\mathbf{M} \geq 2)$ is an O- space iff for each point $p \in \mathbf{M}$ and for any two vectors u, v at p there exists a skew-symmetric endomorphism $T_{u,v}$ of $T_p\mathbf{M}$ such that:*

$$(\kappa_{u,v}^2)' = \kappa_{u,v}^2 \circ T_{u,v} - T_{u,v} \circ \kappa_{u,v}^2. \tag{2.8}$$

Proof. We notice that if (2.8) holds for orthonormal u, v then it holds for arbitrary u, v. Let $(\mathbf{M}$,g) be an O-space and u, v orthonormal vectors at a point $p \in \mathbf{M}$. Let $c(t)$ be the unit circle on $\mathbf{M}$ determined by the initial conditions (2.3) and $A(t)$ the self-adjoint endomorphism of $T_p\mathbf{M}$ which is obtained by a parallel translation of κ_c^2 along c from $c(t)$ to p. Applying Lemma 4 of [1] to the operator $A(t)$, we obtain the endomorphism $T_{u,v}$ of $T_p\mathbf{M}$.

To prove the converse, let $c(t)$ be a unit circle determined by (2.3). Let $A(t)$ be defined as above and let $T(t)$ be the skew- symmetric endomorphism of $T_p\mathbf{M}$ which is obtained by parallel translation of $T_{\dot{c}, \nabla_{\dot{c}}\dot{c}}$ along c from the point $c(t)$ to the point p. Then $A' = A \circ T - T \circ A$ and Lemma 4 of [1] implies that the eigenvalues of A, therefore of κ_c^2, are constant. By Proposition 2.2, $\mathbf{M}$ is an O-space. **Q.E.D.**

Further we have

Theorem 2.4 *Let $(\mathbf{M},g)$ be a smooth Riemannian manifold of dimension $n (n \geq 2)$. Then the following conditions are equivalent:*

i) $\mathbf{M}$ is an O-space;

ii) For each point $p \in \mathbf{M}$ and for any two tangent vectors u, v at p there exists a skew-symmetric endomorphism $T_{u,v}$ of $T_p\mathbf{M}$ such that

$$\kappa'_{u,v} = \kappa_{u,v} \circ T_{u,v} - T_{u,v} \circ \kappa_{u,v}. \tag{2.9}$$

Proof. To prove i) $\rightarrow$ ii) we need the following

Lemma 2.5 *Let $c(t)$ be a unit circle and κ_t be a smooth family of skew-symmetric linear operators along $c(t)$. If the eigenvalues of κ_t are constant then there exists locally a smooth Jordanian basis for κ_t along $c(t)$.*

Proof of the Lemma 2.5. By the conditions of the lemma the eigenvalues of the family of self-adjoint operators κ_t^2 are also constant. Then κ_t^2 is locally C^k-diagonalizable, say $\kappa_t(E_i(t)) = -a_i^2 E_i(t), a_i \in \mathbf{R}, i = 1, ..., l$. Then by the standard procedure we obtain locally a smooth Jordanian basis $\mathcal{B}$ for κ_t, which proves the lemma.

Let $\mathbf{M}$ be an O-space and u, v tangent vectors at a point $p \in \mathbf{M}$. Without loss of generality we may assume that u, v are orthonormal. Let $c(t), t \in I$, be the unit circle determined by the initial conditions (2.3). By Lemma 2.5 we can find locally a smooth Jordanian basis $\mathcal{B}$ of κ_c along the circle $c(t)$. We consider the family θ_t of skew-symmetric linear operators along $c(t)$, defined by $\theta_t(w) := (\nabla_{\dot{c}} w)_t$, for $w \in \mathcal{B}, t \in J \subset I$. We shall prove that for each $w \in \mathcal{B}$ and for each $t \in J$ the following equality holds:

$$(\nabla_{\dot{c}} R)(\dot{c}, \nabla_{\dot{c}}\dot{c})w = \theta(R(\dot{c}, \nabla_{\dot{c}}\dot{c})w) - R(\dot{c}, \nabla_{\dot{c}}\dot{c})\theta(w). \tag{2.10}$$

Using (2.2), we obtain from (2.4) the equality

$$\kappa_c'(w) = \nabla_{\dot{c}}(\kappa_c w) - \kappa_c \theta(w). \tag{2.11}$$

It is easy to verify that the following equality holds:

$$\nabla_{\dot{c}}(\kappa_c w) = \theta(\kappa_c(w)). \tag{2.12}$$

Now, (2.10) follows from (2.11) and (2.12). Set $T_{u,v}(t) := \theta_t$. Then (2.10) implies (2.9).

For the converse, if ii) holds then it is easy to check that (2.8) also holds. Then Proposition 2.3 implies i) which completes the proof of the theorem. **Q.E.D.**

As an immediate consequence of Theorem 2.4 we get

Corollary 2.6 *If* ($\mathbf{M}$,g) *admits an* $(1, 2)$ *tensor field* T *written as* $T_X Z$*, skew-symmetric with respect to the metric* g *and such that*

$$(\nabla_X R)(X, Y)Z = T_X R(X, Y)Z - R(X, Y)T_X Z$$

for all tangent vectors X, Y, Z *at every point of* $\mathbf{M}$*, then* ($\mathbf{M}$,g) *is an O-space.*

Corollary 2.7 *Let* $\mathbf{M} = \mathbf{M}_1 \times ... \times \mathbf{M}_r$ *be a locally reducible smooth Riemannian manifold and each of* $\mathbf{M}_i$ *is smooth. Then* $\mathbf{M}$ *is an O-space iff each of* $\mathbf{M}_i$ *is an O-space.*

3 Two and three dimensional O-spaces

In this section we treat the classification problem for O-spaces of dimension two and three.

For the two dimensional case the eigenvalues of κ_c are $\sqrt{-1}s$ and $-\sqrt{-1}s$, where s is the scalar curvature. Thus we have

Proposition 3.1 *A connected two-dimensional smooth Riemannian manifold is an O-space iff it is of constant sectional curvature.*

The key point for the three dimensional case is the following

Proposition 3.2 *Let* ($\mathbf{M}$,g) *be a 3-dimensional Riemannian manifold. Then* $\mathbf{M}$ *is an O-space iff the following identity holds*

$$\begin{gathered} R(X,Y,Y,X)(\nabla_X R)(X,Y,Y,X) + R(X,Y,Y,Z)(\nabla_X R)(X,Y,Y,Z) + \\ R(Y,X,X,Z)(\nabla_X R)(Y,X,X,Z) = 0 \end{gathered} \tag{3.13}$$

for any orthonormal vectors X, Y, Z *at any point* $p \in \mathbf{M}$.

Proof. Since $\mathbf{M}$ is 3-dimensional then $\mathbf{M}$ is an O-space iff for any unit circle $c(t)$

$$\operatorname{trace}(\kappa_c^2) = \text{const}. \tag{3.14}$$

Let $c(t)$ be the unit circle determined by the initial conditions (2.3), n its first normal given by(2.2) and b its second normal. The second normal b is parallel vector field along $c(t)$ and $\dot{c}, n, b$ form an orthonormal basis along the circle $c(t)$. The equality (3.14) is equivalent to the following equation

$$\dot{c}\left[g(R(\dot{c},n)\dot{c}, R(\dot{c},n)\dot{c}) + g(R(\dot{c},n)n, R(\dot{c},n)n) + g(R(\dot{c},n)b, R(\dot{c},n)b)\right] = 0.$$

By (2.1) the latter equality can be written as

$$g((\nabla_{\dot{c}}R)(\dot{c},n)\dot{c}, R(\dot{c},n)\dot{c}) + g((\nabla_{\dot{c}}R)(\dot{c},n)n, R(\dot{c},n)n) +$$

$$g((\nabla_{\dot{c}}R)(\dot{c},n)b, R(\dot{c},n)b) = 0.$$

Evaluating this equality at the point p we obtain (3.13). **Q.E.D.**

In every point $p \in \mathbf{M}$ we consider the Ricci operator Ric as a linear self-adjoint operator on the tangential space $T_p\mathbf{M}$. Let Ω be the subset of $\mathbf{M}$ on which the number of distinct eigenvalues of Ric is locally constant. This set is open and dense on $\mathbf{M}$. We can choose C^∞ eigenvalue functions of Ric on Ω, say r_1, r_2, r_3, in such a way such that they form the spectrum of Ric at each point of Ω (see for example [1, 9, 5]). We fix a point $p \in \Omega$. Then there exists a local orthonormal frame field E_1, E_2, E_3 on an open connected neighborhood U of p such that

$$Ric(E_i) = r_i E_i, \qquad i = 1,2,3. \tag{3.15}$$

We set $\omega_{ij,k} = g(\nabla_{E_i}E_j, E_k), \quad i,j,k \in \{1,2,3\}$.
We obtain from Proposition 3.2 the following technical result

Proposition 3.3 *Let $(\mathbf{M},g)$ be a 3-dimensional Riemannian manifold. Let Ω be an open and dense set of $\mathbf{M}$ and r_1, r_2, r_3 smooth functions forming at each point of Ω the spectrum of Ric. Let $U \subset \Omega$ and let E_1, E_2, E_3 be orthonormal vector fields on U satisfying the condition (3.15). If $\mathbf{M}$ is an O-space then, for all distinct $i,j,k \in \{1,2,3\}$,the following formulae are valid on U:*

$$E_i(r_i) = 0, \qquad E_i(r_j - r_k) = 0, \tag{3.16}$$

$$E_i(r_j(r_k - r_i)) = -2r_j(r_k - r_i)\omega_{kk,i} \quad , \tag{3.17}$$

$$r_j(r_k - r_i)\omega_{kk,i} = r_k(r_j - r_i)\omega_{jj,i} \quad , \tag{3.18}$$

$$r_i(r_j - r_k)\omega_{ij,k} = 0. \tag{3.19}$$

Proof. For a 3-dimensional Riemannian, manifold the curvature tensor R is given by

$$R(X,Y)Z = ric(Y,Z)X - ric(X,Z)Y + g(Y,Z)Ric(X) - g(X,Z)Ric(Y) - \tag{3.20}$$

$$-\frac{1}{2}s\left[g(Y,Z)X - g(X,Z)Y\right], \quad X,Y,Z \in T_p\mathbf{M}, p \in \mathbf{M},$$

where ric and s denote the Ricci tensor and the scalar curvature of R, respectively.

Using (3.20), we see that (3.13) is equivalent to

$$\begin{aligned}
(3.21)\qquad & [s-2ric(Z,Z)]\,[2(\nabla_X ric)(X,X)+2(\nabla_X ric)(Y,Y)-X(s)]+ \\
& 4ric(X,Z)(\nabla_X ric)(X,Z)+4ric(Y,Z)(\nabla_X ric)(Y,Z)=0,
\end{aligned}$$

for any orthonormal tangent vectors X, Y, Z.

Let $A=(a^1,a^2,a^3), B=(b^1,b^2,b^3), C=(c^1,c^2,c^3)$ be three orthonormal vectors in $\mathbf{R}^3$. We set $X=a^iE_i,\quad Y=b^iE_i,\quad Z=c^iE_i$ (the Einstein summation convention is assumed). Substituting these vectors into (3.21), we obtain after some calculations

$$\begin{aligned}
(3.22)\qquad & \Big[r_1+(2(c^3)^2-1)r_2+(2(c^2)^2-1)r_3\Big]\,X(r_1)+ \\
& \Big[r_2+(2(c^3)^2-1)r_1+(2(c^1)^2-1)r_3\Big]\,X(r_2)+ \\
& \Big[r_3+(2(c^2)^2-1)r_1+(2(c^1)^2-1)r_2\Big]\,X(r_3)- \\
& 4a^1c^1c^2r_3(\nabla_{E_1}ric)(E_1,E_2)-4a^2c^1c^2r_3(\nabla_{E_2}ric)(E_2,E_1)- \\
& 4a^1c^1c^3r_2(\nabla_{E_1}ric)(E_1,E_3)-4a^3c^1c^3r_2(\nabla_{E_3}ric)(E_3,E_1)- \\
& 4a^2c^2c^3r_1(\nabla_{E_2}ric)(E_2,E_3)-4a^3c^3c^2r_1(\nabla_{E_3}ric)(E_3,E_2)- \\
& 4a^1c^2c^3r_1(\nabla_{E_1}ric)(E_2,E_3)-4a^2c^3c^1r_2(\nabla_{E_2}ric)(E_3,E_1)- \\
& 4a^3c^1c^2r_3(\nabla_{E_3}ric)(E_1,E_2)=0.
\end{aligned}$$

If we take $A=(1,0,0), C=(0,1,0)$ then (3.22) implies

$$(3.23)\qquad E_1(r_1-r_2+r_3)=0.$$

Choosing $A=(1,0,0), C=(0,0,1)$ we get from (3.22) that

$$(3.24)\qquad E_1(r_1+r_2-r_3)=0.$$

The equalities (3.23) and (3.24) imply $E_1(r_1)=E_1(r_2-r_3)=0$. Similarly we get all other equalities in (3.16).

Further, we take $A=(1,0,0), C=(0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$. Using (3.16) and (3.22) we get $r_1(\nabla_{E_1}ric)(E_2,E_3)=0$. We obtain (3.19) in an analogous way.

Since the vectors $A(a^1,a^2,a^3)$, $B(b^1,b^2,b^3)$, $C(c^1,c^2,c^3)$ are orthonormal, the vectors $A'=(-a^1,a^2,a^3)$, $B'=(-b^1,b^2,b^3)$, $C'=(-c^1,c^2,c^3)$ are also orthonormal vectors. We take consequently $a^1=\frac{\sqrt{2}}{2}, a^2=0, a^3=\frac{\sqrt{2}}{2},\quad c^1=\frac{\sqrt{2}}{2}, c^2=0, c^3=-\frac{\sqrt{2}}{2}$ and $a^1=0, a^2=\frac{\sqrt{2}}{2}, a^3=\frac{\sqrt{2}}{2},\quad c^1=\frac{\sqrt{3}}{3}, c^2=\frac{\sqrt{3}}{3}, c^3=-\frac{\sqrt{3}}{3}$. Replacing A, B, C by A', B', C' in (3.22), subtracting the last equality from (3.22) and taking into account (3.16) and (3.19) we derive

$$-(-r_1+r_2+r_3)E_1(r_2)=2r_2(\nabla_{E_3}ric)(E_3,E_1)=2r_3(\nabla_{E_2}ric)(E_2,E_1).$$

Similarly, for distinct $i, j, k=1,2,3$ we have

$$-(-r_i+r_j+r_k)E_i(r_j)=2r_j(\nabla_{E_k}ric)(E_k,E_i)=2r_k(\nabla_{E_j}ric)(E_j,E_i).$$

The latter equations are equivalent to the equations:

$$-(-r_i+r_j+r_k)E_i(r_j)=2r_j(r_k-r_i)\omega_{kk,i}=2r_k(r_j-r_i)\omega_{jj,i}$$

for distinct $i, j, k=1,2,3$. Taking into account (3.16), we get (3.17) and (3.18). **Q.E.D.**

4 Proof of Theorem 1.1

Let Ω be an open and dense set in $\mathbf{M}$ and r_1, r_2, r_3 smooth functions on Ω representing the spectrum of the Ricci operator. Let Ω^* be the set of all points p of Ω where each of the functions $r_i, \quad r_i - r_j, \quad i,j = 1,2,3, i \neq j$ (separately) is either identically zero in a neighborhood of p or nonzero in a whole neighborhood. Then Ω^* is open and dense in $\mathbf{M}$.

Let $p \in \mathbf{M}$ and let $U \subset \Omega^*$ be a neighborhood of p. We can find smooth orthonormal vector fields E_1, E_2, E_3 on U satisfying (3.15).We have the following possibilities for the functions $r_1(r_2 - r_3), \quad r_2(r_3 - r_1), \quad r_3(r_1 - r_2)$ on U:

Case A. All of them are identically equal to zero;

Case B. One of them is identically equal to zero and the others are non zero everywhere on U;

Case C. All of them are non zero everywhere on U.

CASE A: In this case we have (up to ordering) two possibilities for the functions r_1, r_2, r_3:

$$r_1 = r_2 = r_3 \qquad \text{and} \qquad r_1 = r_2 = 0, r_3 \neq 0.$$

In the first case (U, g) is an Einstein manifold and hence it is of constant sectional curvature. In the second case we get $r_3 = const. \neq 0$ from (3.16).

CASE B: Let us assume

$$r_1(r_2 - r_3) = 0, \quad r_2(r_3 - r_1) \neq 0, \quad r_3(r_1 - r_2) \neq 0. \tag{4.25}$$

Using (4.25), we get from (3.18) and (3.19) that

$$\omega_{33,1} = \omega_{22,1} \quad , \quad \omega_{23,1} = \omega_{32,1} = \omega_{11,2} = \omega_{11,3} = 0. \tag{4.26}$$

Taking into account (4.26) we calculate

$$0 = ric(E_2, E_1) = g(R(E_2, E_3)E_3, E_1) = E_2(\omega_{33,1}), \tag{4.27}$$

$$0 = ric(E_3, E_1) = g(R(E_3, E_2)E_2, E_1) = E_3(\omega_{22,1}).$$

Because of (4.26) and (4.27), we can apply a result of Hiepko [3] to obtain that U is locally a warped product of type $N_1 \times_\phi N_2$ with an 1-dimensional base N_1 and a 2-dimensional leaf N_2 where $\phi > 0$ is a smooth function on N_1. Let us denote by π and σ the natural projections $\pi : N_1 \times N_2 \to N_1$ and $\sigma : N_1 \times N_2 \to N_2$. Then we have

$$g = \pi^* g^{N_1} + (\pi^* \phi)^2 \sigma^* g^{N_2},$$

where g^{N_1} and g^{N_2} are the metric on N_1 and N_2, respectively. By a simple calculations we obtain

$$r_1 = -2\frac{E_1^2(\pi^*\phi)}{\pi^*\phi}, \quad r_2 = r_3 = \frac{\pi^* k}{(\pi^*\phi)^2} - \frac{(E_1(\pi^*\phi))^2}{(\pi^*\phi)^2} - \frac{E_1^2(\pi^*\phi)}{\pi^*\phi}, \tag{4.28}$$

where k is the sectional curvature of N_2 with respect to g^{N_2}. Using (4.28) we derive from(3.16) and (3.17) that

$$r_1 = \text{const.}; \qquad E_1(r_3(r_3 - r_1)) = -2r_3(r_3 - r_1)\omega_{33,1} \tag{4.29}$$

Further, using (4.29) and (4.28) we calculate

$$E_1(r_3(r_3-r_1)) = (2r_3-r_1)E_1(r_3) = 2(r_3-r_1)(2r_3-r_1)\omega_{33,1}. \tag{4.30}$$

By (4.25), $r_3 - r_1 \neq 0$ on U. Let $p \in U$ and $\omega_{33,1}(p) \neq 0$. Then there exists a neighborhood of p, say $W \subset U$, such that $\omega_{33,1} \neq 0$ everywhere on W. The equalities (4.30) and (4.29) imply $r_3 = \frac{1}{3}r_1 =$ const. Then (4.29) implies $r_3 = r_1 = 0$ on W, but this is impossible because of (4.25). Hence, $\omega_{33,1} = 0$ on U. Analogously, $\omega_{22,1} = 0$ on U. Therefore $\mathbf{M} = N_1 \times N_2$ (see [3], p.209). Since $\mathbf{M}$ is an O-space then N_2 is of constant sectional curvature by of Proposition 3.1 and Corollary 2.7.

CASE C: We may assume

$$r_i(r_j - r_k) \neq 0, \tag{4.31}$$

for distinct $i,j,k, \in \{1,2,3\}$. We get $\omega_{ij,k} = 0$ for distinct $i,j,k = 1,2,3$ from (3.19). Then we can apply Lemma 8 of [1] to conclude that there is a local coordinate system (U,x) of $\mathbf{M}$ and positive functions $\mu_i(x_1,x_2,x_3), i = 1,2,3$ such that the metric g is given in the form

$$g = \mu_1^2(x_1,x_2,x_3)dx_1^2 + \mu_2^2(x_1,x_2,x_3)dx_2^2 + \mu_3^2(x_1,x_2,x_3)dx_3^2. \tag{4.32}$$

and the coordinate vectors $X_i := \frac{\partial}{\partial x_i}$ are eigenvectors of the Ricci tensor everywhere on U. We may assume $X_i = \mu_i E_i, i = 1,2,3$.
Following [1] we set

$$\nu_s = ln(\mu_s), \quad \nu_{s,t} = \frac{\partial \nu_s}{\partial x_t}, \quad \nu_{s,tq} = \frac{\partial^2 \nu_s}{\partial x_t \partial x_q}, \quad Ric(X_t) = \sum_{q=1}^{3} S_t^q x_q, \quad s,t,q = 1,2,3$$

Then for distinct i,j,k we have (see [1]).

$$\begin{aligned} S_i^i = {} & \frac{1}{\mu_j^2}\left[\nu_{i,j}\nu_{j,j} - \nu_{i,j}^2 - \nu_{i,jj} - \nu_{i,j}\nu_{k,j}\right] + \\ & \frac{1}{\mu_k^2}\left[\nu_{i,k}\nu_{k,k} - \nu_{i,k}^2 - \nu_{i,kk} - \nu_{i,k}\nu_{j,k}\right] + \\ & \frac{1}{\mu_i^2}\left[\nu_{j,i}\nu_{i,i} - \nu_{j,i}^2 - \nu_{j,ii} + \nu_{k,i}\nu_{i,i} - \nu_{k,i}^2 - \nu_{k,ii}\right], \end{aligned} \tag{4.33}$$

$$\nu_{i,jk} = \nu_{i,j}\nu_{j,k} + \nu_{i,k}\nu_{k,j} - \nu_{i,j}\nu_{i,k} \quad , \tag{4.34}$$

$$\nu_{k,j} + \mu_j\omega_{kk,j} = 0. \tag{4.35}$$

We shall prove that the following system of non-linear partial differential equations holds locally for distinct $i,j,k, = 1,2,3$

$$\begin{aligned} &\nu_{i,jk} = 0, \\ &\nu_{i,j}\nu_{i,k} - \nu_{i,j}\nu_{j,k} - \nu_{i,k}\nu_{k,j} = 0, \\ &\nu_{i,ij} + 2\nu_{i,j}\nu_{j,k} = 0. \end{aligned} \tag{4.36}$$

We get from (4.35) and (3.18) that

$$r_i(r_k - r_j)\nu_{k,j} = r_k(r_i - r_j)\nu_{i,j} \tag{4.37}$$

for distinct i, j, k.
Let V be the set of all points p of U where each of the functions $\nu_{i,j}, i,j = 1,2,3, i \neq j$ is either identically zero in a neighborhood of p or nonzero in a whole neighborhood. Then V is open and dense in U. If $\nu_{i,j} = 0$ around a point $p \in V$, then $\nu_{i,ij} = \nu_{i,ji} = 0$ and it follows from (4.31) and (4.37) that the last two equations of (4.36) are fulfilled. The first equation of (4.36) follows from (4.34) and the second equation of (4.36).

Now, let $\nu_{i,j} \neq 0$ around a point $p \in V$. Then, from (4.31) and (4.37), we get

$$\frac{\nu_{i,j}}{\nu_{k,j}} = \frac{r_i(r_k - r_j)}{r_k(r_i - r_j)} = 1 - \frac{r_j(r_k - r_i)}{r_k(r_j - r_i)}. \tag{4.38}$$

Replacing j by k in (4.37) we get $r_i(r_j - r_k)\nu_{j,k} = r_j(r_i - r_k)\nu_{i,k}$. Then we obtain

$$r_i(r_j - r_k)(\nu_{j,k} - \nu_{i,k}) + r_k(r_j - r_i)\nu_{i,k} = 0. \tag{4.39}$$

Using (4.38), (4.31) and (4.39) we see that the second equation of (4.36) is valid everywhere on V. Then (4.34) and the second equation of (4.36) imply the first equation of (4.36). By (3.16), (3.17) and (4.38) we obtain:

$$\frac{\nu_{i,j}}{\nu_{k,j}} X_i(r_j(r_k - r_i)) = -X_i\left[\frac{\nu_{i,j}}{\nu_{k,j}}\right] r_k(r_j - r_i). \tag{4.40}$$

Using (3.17), (4.35) and (4.37), we get from (4.40) that

$$2\frac{\nu_{i,j}}{\nu_{k,j}}\nu_{j,i} = -X_i\left[\frac{\nu_{i,j}}{\nu_{k,j}}\right].$$

The latter equation and the first equation of (4.36) imply the third equation of (4.36).

The system (4.36) is well known: it arises from a quantum mechanical problems; on the other hand it is shown in [1] that this system describes 3-dimensional P-spaces with distinct Ricci eigenvalues. The complete solution of this system has been given by L.P.Eisenhart in [2]. He has obtained four kinds of solutions, namely:
$(E1)$. $\mu_1^2 = 1, \mu_2^2 = G(x_2)\eta(x_1)(\phi(x_2) + \psi(x_3)); \mu_3^2 = H(x_3)\eta(x_1)(\phi(x_2) + \psi(x_3))$, where η is a function of x_1, G, ϕ are functions of x_2, H, ψ are functions of $x_3 and G, H, \eta > 0$;
$(E2)$. $\mu_1^2 = 1, \quad \mu_2^2 = \phi^2(x_1) \quad \mu_3^2 = \psi^2(x_1)$, where ϕ and ψ are functions of x_1;
$(E3)$. $\mu_1^2 = \mu_2^2 = (\phi(x_1) + \psi(x_2)), \quad \mu_3^2 = |\phi(x_1) + \psi(x_2)|$, where ϕ is a function of x_1 and ψ is a function of x_2.
$(E4)$ $\mu_i^2 = F_i(x_i)|x_i - x_j||x_i - x_k|$, for distinct $i, j, k = 1,2,3$, where F_i are functions of x_i.

To complete the proof of Theorem 1.1 we show below that any metric of the form (4.32) with anyone of the solutions $E1, E2, E3, E4$ does not appear as a metric of an O-space.

CASE $E1$. In this case it is proved in [1] that U is precisely the warped product of an 1-dimensional base and a 2- dimensional leaf. Hence, the number of distinct eigenvalues of the Ricci operator Ric is at most two, which contradicts to (4.31).

CASE $E2$. In this case we have $\nu_1 = \nu_{1,1} = \nu_{1,2} = \nu_{1,3} = \nu_{2,2} = \nu_{2,3} = \nu_{3,2} = \nu_{3,3} = 0$. Using (4.33) we calculate

$$r_1 = S_1^1 = -\nu_{2,1}^2 - \nu_{2,11} - \nu_{3,1}^2 - \nu_{3,11},$$

$$r_2 = S_2^2 = -\nu_{2,1}^2 - \nu_{2,11} - \nu_{2,1}\nu_{3,1}, \tag{4.41}$$

$$r_3 = S^3_3 = -\nu^2_{3,1} - \nu_{3,11} - \nu_{3,1}\nu_{2,1}.$$

Using (3.16) we obtain from (4.41)

$$\nu^2_{2,1} + \nu_{2,11} = B = const., \quad \nu^2_{3,1} + \nu_{3,11} = C = const. \tag{4.42}$$

Now, (4.41) takes the form:

$$r_1 = -B - C, \quad r_2 = -B - \nu_{2,1}\nu_{3,1}, \quad r_3 = -C - \nu_{2,1}\nu_{3,1}. \tag{4.43}$$

The conditions (4.31) imply

$$B \neq C, \qquad B \neq -C. \tag{4.44}$$

We obtain from (3.17) and (4.37) that

$$(\nu_{2,1}\nu_{3,1})^2(\nu_{3,1} - \nu_{2,1}) = B^2\nu_{3,1} - C^2\nu_{2,1}. \tag{4.45}$$

$$X_1(\nu_{2,1}\nu_{3,1})^2 = 2((\nu_{2,1}\nu_{3,1})^2 - B^2)\nu_{3,1}. \tag{4.46}$$

We claim that

$$\nu_{2,1} - \nu_{3,1} \neq 0, \quad \nu_{2,1} \neq 0, \quad \nu_{3,1} \neq 0. \tag{4.47}$$

Indeed, if $\nu_{2,1} = 0$ at a point of V, then the equality (4.42) implies $B = 0$ and, from (4.43) we get $r_2 = 0$ which contradicts to (4.31). Similarly we prove that $\nu_{3,1} \neq 0$. Further, if $\nu_{2,1} = \nu_{3,1}$ at a point of V then (4.45) implies $B^2 = C^2$, which contradicts to (4.44). Keeping in mind (4.47) we get from (4.45) and (4.46):

$$(\nu_{2,1}\nu_{3,1})^2 = \frac{B^2\nu_{3,1} - C^2\nu_{2,1}}{\nu_{3,1} - \nu_{2,1}}. \tag{4.48}$$

$$(\nu_{2,1}\nu_{3,1})' = \frac{B^2 - C^2}{\nu_{3,1} - \nu_{2,1}}. \tag{4.49}$$

Differentiating (4.48) and using (4.49) and (4.42) we get:

$$\nu_{2,1}\nu_{3,1} = \frac{B\nu_{3,1} - C\nu_{2,1}}{\nu_{3,1} - \nu_{2,1}}. \tag{4.50}$$

¿From (4.48), (4.49) and (4.50), keeping in mind (4.47), we get $B^2 = C^2$ which is impossible because of (4.44).

*CASE E*3. In this case we have (see the calculations in [1]):

$$\nu_{2,1} = \frac{\phi'}{2(\phi + \psi)}, \quad \nu_{3,1} = \frac{\phi'}{2\phi}, \quad \nu_{2,2} = \frac{\psi'}{2(\phi + \psi)}, \quad \nu_{3,2} = \frac{\psi'}{2\psi)}. \tag{4.51}$$

$$r_2 - r_3 = \psi T; \quad r_3 - r_1 = \phi T, \quad r_1 - r_2 = -(\phi + \psi)T,$$

where

$$T = \frac{1}{4(\phi + \psi)^3}\left\{\frac{1}{\phi}\left[2\phi''(\phi + \psi) - \frac{(\phi')^2}{\phi}(3\phi + \psi)\right] - \frac{1}{\psi}\left[2\psi''(\phi + \psi) - \frac{(\psi')^2}{\psi}(3\psi + \phi)\right]\right\}$$

In any point of V at least one of the functions ϕ' or ψ' is different from zero (if $\phi' = \psi' = 0$, then $T = 0$ and $r_1 = r_2 = r_3$, which is impossible because of (4.31)).
Let $p \in V$ and $\phi'(p) \neq 0$. Then, by (4.51), we have at p:

$$\frac{\nu_{2,1}}{\nu_{3,1}} = \frac{\phi}{\phi + \psi} = \frac{r_3 - r_1}{r_2 - r_1} \neq 0.$$

Comparing this with (4.38) we get $r_2 = r_3$ at p, which contradicts to (4.31).

ĈASE E4. In this case we have (see [1],p.78) for all distinct $i, j \in \{1, 2, 3\}$

$$\nu_{i,j} = \frac{1}{x_j - x_i}, \quad r_i - r_j = (x_i - x_j)T, \tag{4.52}$$

$$T = \frac{\sigma_{1,2,3}\left(\frac{2}{F_1}\left(\frac{1}{x_1 - x_3} + \frac{1}{x_1 - x_2}\right) - \left(\frac{1}{F_1}\right)'\right)(x_2 - x_3)^2}{4(x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2}.$$

If at some point $p \in V$ we have $T(p) = 0$, then $r_i = r_j$ which is impossible on V.
Let $T(p) \neq 0$. We derive from (4.52) that

$$\frac{r_k - r_i}{r_j - r_i} = \frac{\nu_{j,i}}{\nu_{k,i}}.$$

Similar arguments as in the case E3 show that the case E4 is also impossible.

It is easy to verify that any manifold of type i) ii) and iii) is an O-space. Especially, in the case iii) we check that $r_1 = r_2 = 0$ and $r_3 = const$ always satisfies the equation (3.22), which is equivalent to (3.13). This completes the proof of Theorem 1.1. **Q.E.D.**

Remark 2. It follows from the proof of Theorem 7 in [1] that if a 3-dimensional $\mathcal{P}$-space $\mathbf{M}$ has two distinct Ricci eigenvalues then it is locally isometric to a warped product $N_1 \times_\phi N_2$ with an 1-dimensional base N_1 and a 2-dimensional leaf N_2 where $\phi > 0$ is a smooth function on N_1. If moreover, we assume that $\mathbf{M}$ is also an $\mathcal{O}$-space and $r_1 = const, r_2 = r_3$ it follows from the proof of Theorem 1.1 that $\mathbf{M}$ is locally isometric to the Riemannian product $N_1 \times N_2$ and by Theorem 1.1 $\mathbf{M}$ has to be a locally symmetric Riemannian manifold. Hence, the $\mathcal{O}$-space described in Example 1 is not a $\mathcal{P}$-space.

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References

[1] J.Berndt, L.Vanhecke, *Two natural generalizations of locally symmetric spaces.* J.Diff.Geom. and Appl. $\mathbf{2}(1992), 57-80$.

[2] L.P.Eisenhart, *Separabel system of Stackel*, Ann. of Math. $(2)\mathbf{35}(1934), 284-305$.

[3] S.Hiepko, *Eine innere kennzeichnung der verzerrten producte*, Math. Ann. $\mathbf{241}(1979), 209-215$.

[4] S.Ivanov, I.Petrova, *Curvature operator with parallel Jordanian basis on circles*, Riv. Mat. Univ. Parma, to appear.

[5] O.Kowalski, *A classification of Riemannian manifolds with constant principal Ricci curvatures* $r_1 = r_2 \neq r_3$*.*, Nagoya Math. J. **132**(1993), 1 − 36.

[6] O.Kowalski, F.Prüfer *On Riemannian 3-manifolds with distinct constant Ricci eigenvalues*, Math. Ann. **300**(1994), 17 − 28.

[7] J.Milnor, *Curvature of left-invariant metrics on Lie groups.*, Adv. in Math. **21**(1976), 163 − 170.

[8] K.Nomizu, K.Yano, *On circles and spheres in Riemannian geometry*, Math. Ann. **210**(1974), 163 − 170.

[9] Z.Shabó, *Structure theorems on Riemannian spaces satisfying* $R(X,Y) \circ R = 0$ *,I. The local version*, J.Diff.Geom. **17**(1982)531 − 582.

[10] I.M.Singer, *Infinitesimally homogeneous spaces*, Commun. Pure Appl. Math **13**(1960), 685 − 697.

[11] A.Spiro, F.Tricerri, *3-dimensional Riemannian metrics with prescribed Ricci principal curvatures*, J. Math. Pures Appl. **74**(1995), 253 − 271.

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