Пресмятане на определени интеграли

I. *Решете интегралите.*

$$1.\int_{-1}^{8}\sqrt[3]{x}\,dx;\quad 2.\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}tg^2x\,dx;\quad 3.\int_{0}^{\frac{1}{2}}\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}\,dx;$$

$$4.\int_{0}^{1}\frac{x}{(1+x^2)^2}\,dx;\quad 5.\int_{1}^{e}\frac{1+\ln x}{x}\,dx;\quad 6.\int_{1}^{e}\frac{dx}{x\sqrt{1-\ln^2 x}};$$

$$7.\int_{1}^{2}\frac{e^{1/x}}{x^2}\,dx;\quad 8.\int_{2}^{3}\frac{dx}{2x^2+3x-2};\quad 9.\int_{0}^{3}\frac{dx}{3+10x+8x^2};$$

$$10.\int_{1}^{2}\frac{dx}{x+x^3};\quad 11.\int_{-0.5}^{1}\frac{dx}{\sqrt{8+2x-x^2}}.$$

Решения:

$$1).\ I_1=\int_{-1}^{8}x^{1/3}\,dx=\tfrac{3}{4}x^{4/3}\Big|_{-1}^{8}=\tfrac{3}{4}(8^{4/3}-(-1)^{4/3})=\tfrac{3}{4}(16-1)=\tfrac{45}{4};$$

$$2).\ I_2=\int_{-\pi/4}^{\pi/4}\frac{\sin^2x-1}{\cos^2x}\,dx+\int_{-\pi/4}^{\pi/4}\frac{dx}{\cos^2x}=-\int_{-\pi/4}^{\pi/4}\frac{\cos^2x}{\cos^2x}\,dx+\int_{-\pi/4}^{\pi/4}\frac{dx}{\cos^2x}=$$

$$=-\int_{-\pi/4}^{\pi/4}dx+tgx\Big|_{-\pi/4}^{\pi/4}=-\tfrac{\pi}{2}+(1-(-1))=2-\tfrac{\pi}{2}=\tfrac{4-\pi}{2};$$

$$3).\ I_3=\int_{0}^{1/2}\frac{dx}{\sqrt{1-x^2}}+\int_{0}^{1/2}\frac{dx}{\sqrt{1+x^2}}=\arcsin\Big|_{0}^{1/2}+\ln(x+\sqrt{1+x^2})\Big|_{0}^{1/2}=$$

$$=\tfrac{\pi}{6}+\ln(\tfrac{1}{2}+\tfrac{\sqrt{5}}{2})-\ln 1=\tfrac{\pi}{6}+\ln\tfrac{1+\sqrt{5}}{2};$$

$$4).\ I_4=\tfrac{1}{2}\int_0^1\frac{dx^2}{(1+x^2)^2}=\tfrac{1}{2}\int_0^1\frac{d(x^2+1)}{(1+x^2)^2}=\tfrac{1}{2}(-1)\frac{1}{1+x^2}\Big|_0^1=-\tfrac{1}{2}(\tfrac{1}{2}-1)=\tfrac{1}{4};$$

$$5).\ I_5=\int_1^e\tfrac{1}{x}\,dx+\int_1^e\tfrac{\ln x}{x}\,dx=\ln x\Big|_1^e+\int_1^e\ln x\,d(\ln x)=\ln x\Big|_1^e+\tfrac{1}{2}\ln^2 x\Big|_1^e=$$
$$=1-0+\tfrac{1}{2}(1-0)=\tfrac{3}{2};$$

$$6).\ I_6=\int_1^e\frac{d(\ln x)}{\sqrt{1-\ln^2 x}}=\arcsin(\ln x)\Big|_1^e=\arcsin(\ln e)-\arcsin(\ln 1)=$$
$$=\arcsin 1-\arcsin 0=\tfrac{\pi}{2}-0=\tfrac{\pi}{2};$$

$$7).\ I_7=-\int_1^2 e^{1/x}d(1/x)=-e^{1/x}\Big|_1^2=-(e^{1/2}-e)=e-\sqrt{e};$$

$8).\ I_8$ = **?:** Преобразуваме подинтегралната функция:

$$\frac{1}{2x^2+3x-2}=\frac{1}{2(x+2)(x-\frac{1}{2})}=\frac{1}{(x+2)(2x-1)}=\frac{a}{x+2}+\frac{b}{2x-1}\Leftrightarrow$$
$$1=a(2x-1)+b(x+2);$$

от последното равенство, при **x = -2** и **x = 1/2,** съответно за ***a*** и ***b***, получаваме: ***a* = -1/5, *b* = 2/5;** тогава:

$$I_8=-\tfrac{1}{5}\int_2^3\frac{dx}{x+2}+\tfrac{2}{5}\int_2^3\frac{dx}{2x-1}=-\tfrac{1}{5}\ln|x+2|\Big|_2^3+\tfrac{1}{5}\ln|2x-1|\Big|_2^3=$$
$$=\tfrac{1}{5}(\ln 4-\ln 5+\ln 5-\ln 3)=\tfrac{1}{5}(\ln 4-\ln 3)=\ln\sqrt[5]{\tfrac{4}{3}};$$

$9).\ I_9$ **= ?:** Преобразуваме подинтегралната функция:

$$\frac{1}{3+10x+8x^2}=\frac{1}{8(x+\frac{3}{4})(x+\frac{1}{2})}=\frac{1}{(4x+3)(2x+1)}=\frac{a}{4x+3}+\frac{b}{2x+1},$$

$$\Leftrightarrow 1=a(2x+1)+b(4x+3),\Rightarrow: a=-2\ /x=-\tfrac{3}{4}/;\ b=1\ /x=-\tfrac{1}{2}/,\Rightarrow$$

$$I_9=-2\int_0^3\frac{dx}{4x+3}+\int_0^3\frac{dx}{2x+1}=-\tfrac{2}{4}\ln|4x+3|\,\big|_0^3+\tfrac{1}{2}\ln|2x+1|\,\big|_0^3=$$

$$=\tfrac{1}{2}(\ln 3-\ln 15+\ln 7-\ln 1)=\tfrac{1}{2}(\ln 3-\ln 15+\ln 7)=\tfrac{1}{2}(\ln 21-\ln 15)=$$

$$=\tfrac{1}{2}\ln\tfrac{21}{15}=\tfrac{1}{2}\ln\tfrac{7}{5}=\ln\sqrt{\tfrac{7}{5}}\ ;$$

10). I_{10} **= ?:** Преобразуваме подинтегралната функция:

$$\frac{1}{x+x^3}=\frac{1}{x(1+x^2)}=\frac{a}{x}+\frac{bx+c}{1+x^2},\Leftrightarrow 1=a(1+x^2)+$$

$$+x(bx+c);\Rightarrow a=1\ \ /x=0/;$$

за по-удобното пресмятане на коефициентите ***b*** и ***c,*** диференцираме тъждеството $1=a(1+x^2)+x(bx+c)$ **:**

$$0=2ax+2bx+c,\Rightarrow c=0\ \ /x=0/,\ b=-1\ \ /x=1/.$$

Сега за интеграла $\boldsymbol{I_{10}}$ получаваме:

$$I_{10}=\int_1^2\frac{dx}{x}-\int_1^2\frac{xdx}{1+x^2}=\ln|x|\,\big|_1^2-\tfrac{1}{2}\int_1^2\frac{d(x^2+1)}{1+x^2}=\ln 2-\ln 1-$$

$$-\tfrac{1}{2}\ln(1+x^2)\,\big|_1^2=\ln 2-\tfrac{1}{2}(\ln 5-\ln 2)=\tfrac{3}{2}\ln 2-\tfrac{1}{2}\ln 5=\tfrac{1}{2}(\ln 2^3-\ln 5)=$$

$$=\tfrac{1}{2}\ln\tfrac{8}{5}=\ln\sqrt{\tfrac{8}{5}}=\ln\tfrac{2\sqrt{2}}{\sqrt{5}}\ ;$$

$$11).\ I_{11}=\int_{-0.5}^{1}\frac{dx}{\sqrt{9-(x-1)^2}}=\int_{-0.5}^{1}\frac{d(x-1)}{\sqrt{9-(x-1)^2}}=$$

$$=\tfrac{1}{3}\int_{-0.5}^{1}\frac{d(x-1)}{\sqrt{1-((x-1)/3)^2}}=\int_{-0.5}^{1}\frac{d((x-1)/3)}{\sqrt{1-((x-1)/3)^2}}=$$

$$=\arcsin\tfrac{x-1}{3}\,\big|_{-0.5}^{1}=0-\arcsin\tfrac{(-1.5)}{3}=\arcsin\tfrac{1}{2}=\tfrac{\pi}{6}.$$

II. *Определени интеграли от произведения (степени) на* **sin** *и* **cos** **.**

$$1^0.\ \int_{-\pi}^{\pi}\cos 3x.\cos 7x\,dx;\quad 2^0.\ \int_{-\pi/2}^{\pi}\sin 2x.\sin 5x\,dx;$$

$$3^0.\ \int_{0}^{\pi/2}\cos^5 x.\sin 2x\,dx;\quad 4^0.\ \int_{-\pi/2}^{\pi/2}\sin^3 x\,dx;\quad 5^0.\ \int_{0}^{\pi/2}\cos^3 2x\,dx;$$

$$6^0.\ \int_{-\pi}^{\pi/2}\sin^3 x.\cos^2 x\,dx.$$

Решения:

$1^0)\,J_1 = ?$**:** Преобразуваме подинтегралната функция **:**

$$\cos 3x.\cos 7x = \tfrac{1}{2}(\cos 10x + \cos 4x).$$

За интеграла $\boldsymbol{J_1}$ получаваме**:**

$$J_1 = \int_{-\pi}^{\pi}\cos 3x.\cos 7x\,dx = \tfrac{1}{2}\int_{-\pi}^{\pi}\cos 10x\,dx + \tfrac{1}{2}\int_{-\pi}^{\pi}\cos 4x\,dx =$$
$$= \tfrac{1}{2.10}\sin 10x\,|_{-\pi}^{\pi} + \tfrac{1}{2.4}\sin 4x\,|_{-\pi}^{\pi} = \tfrac{1}{20}.0 + \tfrac{1}{8}.0 = 0.$$

$$2^0).\,J_2 = \int_{-\pi/2}^{\pi}\sin 2x.\sin 5x\,dx = ?\ \mathbf{:}$$

$$\sin 2x.\sin 5x = \tfrac{1}{2}(\cos 3x - \cos 7x), \Rightarrow$$
$$J_2 = \tfrac{1}{2}\int_{-\pi/2}^{\pi}\cos 3x\,dx - \tfrac{1}{2}\int_{-\pi/2}^{\pi}\cos 7x\,dx = \tfrac{1}{6}\sin 3x\,|_{-\pi/2}^{\pi} -$$
$$-\tfrac{1}{14}\sin 7x\,|_{-\pi/2}^{\pi} = \tfrac{1}{6}(0-(-1)\sin\tfrac{3\pi}{2}) - \tfrac{1}{14}(0-(-1)\sin\tfrac{7\pi}{2})$$
$$= -\tfrac{1}{6} + \tfrac{1}{14} = -\tfrac{2}{21}.$$

$$3^0).\ J_3=\int_0^{\pi/2}\cos^5 x.\sin 2x\,dx=2\int_0^{\pi/2}\cos^6 x.\sin x\,dx=$$

$$=-2\int_0^{\pi/2}\cos^6 x d(\cos x)=-\tfrac{2}{7}\cos^7 x\Big|_0^{\pi/2}=-\tfrac{2}{7}(0-1)=\tfrac{2}{7}.$$

$$4^0).\ J_4=\int_{-\pi/2}^{\pi/2}\sin^3 x\,dx=\int_{-\pi/2}^{\pi/2}\sin^2 x.\sin x\,dx=$$

$$=-\int_{-\pi/2}^{\pi/2}(1-\cos^2 x)\,d(\cos x)=-\cos x\Big|_{-\pi/2}^{\pi/2}+\int_{-\pi/2}^{\pi/2}\cos^2 x\,d(\cos x)=$$

$$=\tfrac{1}{3}\cos^3 x\Big|_{-\pi/2}^{\pi/2}=\tfrac{1}{3}.0=0.$$

$$5^0).\ J_5=\int_0^{\pi/2}\cos^3 2x\,dx=\int_0^{\pi/2}\cos^2 2x.\cos 2x\,dx=$$

$$=\tfrac{1}{2}\int_0^{\pi/2}\cos^2 2x\,d(\sin 2x)=\tfrac{1}{2}\int_0^{\pi/2}(1-\sin^2 2x)\,d(\sin 2x)=$$

$$=\tfrac{1}{2}\sin 2x\Big|_0^{\pi/2}-\tfrac{1}{2}\int_0^{\pi/2}\sin^2 2x\,d(\sin 2x)=-\tfrac{1}{6}\sin^3 2x\Big|_0^{\pi/2}=0.$$

$$6^0).\ J_6=\int_{-\pi}^{\pi/2}\sin^3 x.\cos^2 x\,dx=\int_{-\pi}^{\pi/2}\sin^2 x.\cos^2 x.\sin x\,dx=$$

$$=-\int_{-\pi}^{\pi/2}(1-\cos^2 x)\cos^2 x\,d(\cos x)=-\int_{-\pi}^{\pi/2}\cos^2 x\,d(\cos x)+$$

$$+\int_{-\pi}^{\pi/2}\cos^4 x\,d(\cos x)=-\tfrac{1}{3}\cos^3 x\Big|_{-\pi}^{\pi/2}+\tfrac{1}{5}\cos^5 x\Big|_{-\pi}^{\pi/2}=$$

$$=-\tfrac{1}{3}(0-(-1))+\tfrac{1}{5}(0-(-1))=-\tfrac{1}{3}+\tfrac{1}{5}=-\tfrac{2}{15}.$$

III. Определени интеграли – общи задачи.

$$1^*.\ I_1^* = \int_{-1}^{1}(1-x^2)^3\,dx;\quad 2^*.\ I_2^* = \int_{0}^{1}x^2(1-x)^3\,dx;$$

$$3^*.\ I_3^* = \int_{1}^{e}\ln^3 x\,dx;\quad 4^*.\ I_4^* = \int_{1}^{2}x^3\ln^2 x\,dx;$$

$$5^*.\ I_5^* = \int_{\pi/4}^{\pi/3}\frac{xdx}{\sin^2 x};\quad 6^*.\ I_6^* = \int_{0}^{a}\sqrt{a^2-x^2}\,dx\ \ (a>0)$$

Решения:

$$1^*).\ I_1^* = \int_{-1}^{1}(1-3x^2+3x^4-x^8)\,dx = (x-x^3+\tfrac{3}{5}x^5-\tfrac{1}{9}x^9)\Big|_{-1}^{1}$$

$$=1-1+\tfrac{3}{5}-\tfrac{1}{9}-(1-1(-1)+\tfrac{3}{5}(-1)-\tfrac{1}{9}(-1))=$$

$$=\tfrac{3}{5}-\tfrac{1}{9}-2+\tfrac{3}{5}-\tfrac{1}{9}=\tfrac{6}{5}-\tfrac{2}{9}-2=\tfrac{46}{45};$$

$$2^*).\ I_2^* = \int_{0}^{1}\tfrac{1}{3}(1-x)^3\,d(x^3) = \tfrac{1}{3}x^3(1-x)^3\Big|_0^1 -$$

$$-\tfrac{1}{3}\int_{0}^{1}x^3.3(1-x)^2.(-1)\,dx = 0+\tfrac{3}{12}\int_{0}^{1}(1-x)^2\,d(x^4)=$$

$$=\tfrac{1}{4}x^4(1-x^2)\Big|_0^1 - \tfrac{1}{4}\int_{0}^{1}x^4.2(1-x).(-1)\,dx = 0+$$

$$+\tfrac{1}{2}\int_{0}^{1}x^4(1-x)\,dx = \tfrac{1}{10}\int_{0}^{1}(1-x)\,d(x^5) = \tfrac{1}{10}x^5(1-x)\Big|_0^1 -$$

$$-\tfrac{1}{10}\int_{0}^{1}x^5\,d(1-x) = \tfrac{1}{10}.0+\tfrac{1}{10}\int_{0}^{1}x^5\,dx = \tfrac{1}{60}x^6\Big|_0^1 = \tfrac{1}{60};$$

$$3^*).\ I_3^* = (x\ln^3 x)\Big|_1^e - \int_1^e x.d(\ln^3 x) = e.(\ln e)^3 - 0 -$$

$$-\int_1^e x.3\ln^2 x.\frac{1}{x}\,dx = e - 3\int_1^e \ln^2 x\,dx = e - 3x\ln^2 x\Big|_1^e +$$

$$+3\int_1^e x.2\ln x.\frac{1}{x}\,dx = e - (3e - 0) + 6\int_1^e \ln x\,dx = -2e +$$

$$+6x\ln x\Big|_1^e - 6\int_1^e x.\frac{1}{x}\,dx = -2e + 6e - 0 - 6(e-1) = 6 - 2e;$$

$$4^*).\ I_4^* = \tfrac{1}{4}\int_1^2 \ln^2 x\,d(x^4) = \tfrac{1}{4}x^4\ln^2 x\Big|_1^2 - \tfrac{1}{4}\int_1^2 x^4.2\ln x.\frac{1}{x}\,dx$$

$$= \tfrac{1}{4}(16\ln^2 2 - 0) - \tfrac{1}{2.4}\int_1^2 \ln x\,d(x^4) = 4\ln^2 2 - \tfrac{1}{8}x^4\ln x\Big|_1^2 +$$

$$+\tfrac{1}{8}\int_1^2 x^4.\frac{1}{x}\,dx = 4\ln^2 2 - 2\ln 2 + \tfrac{1}{8.4}x^4\Big|_1^2 = 4\ln^2 2 - 2\ln 2$$

$$+\tfrac{1}{32}(16-1) = 4\ln^2 2 - 2\ln 2 + \tfrac{15}{32};$$

$$5^*).\ I_5^* = -\int_{\pi/4}^{\pi/3} x\,d(ctgx) = -x.ctgx\Big|_{\pi/4}^{\pi/3} + \int_{\pi/4}^{\pi/3} ctgx\,dx =$$

$$= \tfrac{\pi}{4} - \tfrac{\pi}{3\sqrt{3}} + (\ln|\sin x|)\Big|_{\pi/4}^{\pi/3} = \tfrac{\pi}{4} - \tfrac{\pi}{3\sqrt{3}} + \ln\tfrac{\sqrt{3}}{2} - \ln\tfrac{1}{\sqrt{2}} =$$

$$= \tfrac{\pi}{4}(\tfrac{1}{4} - \tfrac{1}{3\sqrt{3}}) + \ln\tfrac{\sqrt{3}}{\sqrt{2}};$$

$$6^*).\ I_6^* = (x\sqrt{a^2-x^2})\Big|_0^a + \int_0^a \frac{x^2dx}{\sqrt{a^2-x^2}} = 0 +$$

$$+\int_0^a \frac{x^2-a^2+a^2}{\sqrt{a^2-x^2}}dx = -\int_0^a \sqrt{a^2-x^2}\,dx + a^2.\int_0^a \frac{dx}{\sqrt{a^2-x^2}}$$

$$= -I_6^* + \tfrac{a^2}{a}\int_0^a \frac{dx}{\sqrt{1-(\frac{x}{a})^2}} = -I_6^* + a^2\int_0^a \frac{d(\frac{x}{a})}{\sqrt{1-(\frac{x}{a})^2}} = -I_6^* +$$

$$+a^2.(\arcsin\tfrac{x}{a})\Big|_0^a = -I_6^* + a^2.\arcsin 1; \Rightarrow$$

$$I_6^* = -I_6^* + a^2.\tfrac{\pi}{2}; \Rightarrow I_6^* = \tfrac{a^2\pi}{4}.$$

Забележка: По същия начин, при ***a < 0*** получаваме **:** $I_6^* = -\frac{a^2\pi}{4}$ **.**